Quantum Verification of Matrix Products

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Matrix multiplication

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- [Coppersmith & Winograd, 1987]
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  Arithmetic progression: time $O(n^{2.376})$
- Best known lower bound is only $\Omega(n^2)$
  The actual complexity is open

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- [Freivalds, 1979]
  Classical algorithm with time $O(n^2)$
  1. Pick a random vector x
  2. Compute $y = Cx$ and $y' = A(Bx)$
  3. Compare y with y'
- Matrix-vector products take time $O(n^2)$
- Constant success probability

Quantum computing

Computers based on laws of quantum physics

- quantum state is a *superposition* of classical states

$$|\psi\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle, \quad \text{where } \alpha_x \in \mathbb{C} \text{ and } \sum_x |\alpha_x|^2 = 1$$

- computational step is defined by

$$|\psi\rangle \to U|\psi\rangle$$

  for a *unitary* (i.e. norm-preserving) operator U

- outcome is observed by a *measurement*
  the probability of seeing x is $|\alpha_x|^2$

$$\Pr[\Psi = x] = |\alpha_x|^2$$

Quantum algorithms for matrix verification

- [Grover, 1996]
  Searching an unsorted database in time $O(\sqrt{n})$
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- [our paper]

Matrix verification in time $O(n^{5/3})$
using two random vectors
and quantum random walks

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$$O\left(T_{\text{init}} + \frac{1}{\sqrt{\delta\varepsilon}} \cdot (T_{\text{test}} + T_{\text{walk}})\right) ,$$

  - T_{init} is time of picking a uniform superposition of vertices
  - T_{test} is time of testing whether a vertex is marked
  - T_{walk} is time of walking one step over G

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  - δ is the spectral gap of G
  - ε is the fraction of marked vertices

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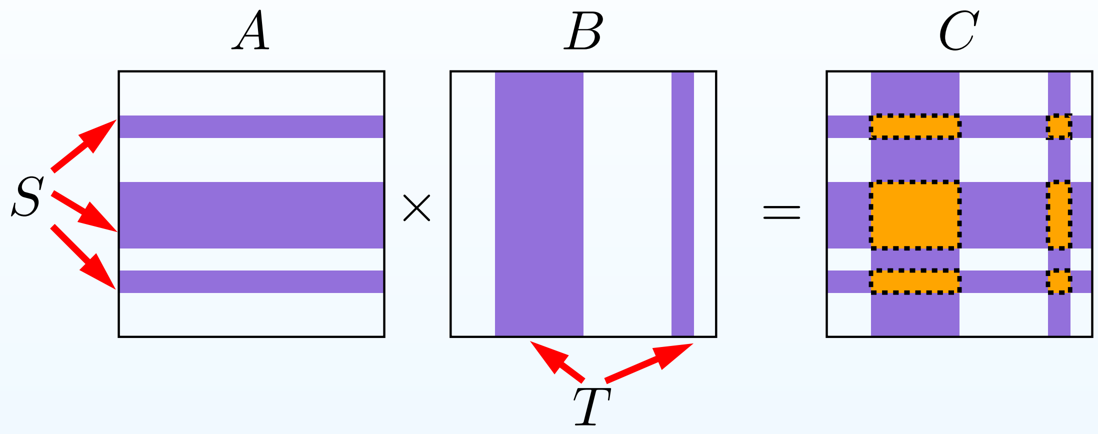
- Classical random walks converge in time proportional to

$$\frac{1}{\delta\varepsilon}$$

Quantum algorithm for matrix verification

Verification of matrix product $AB = C$



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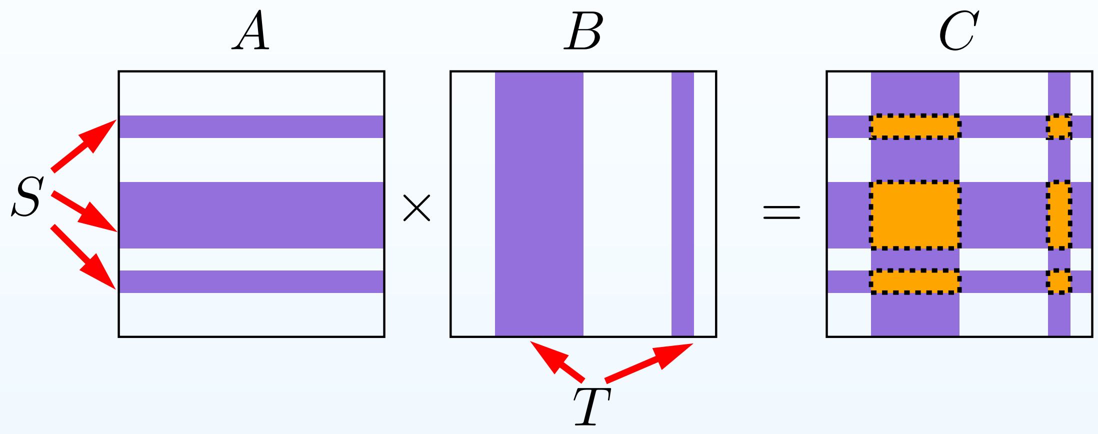

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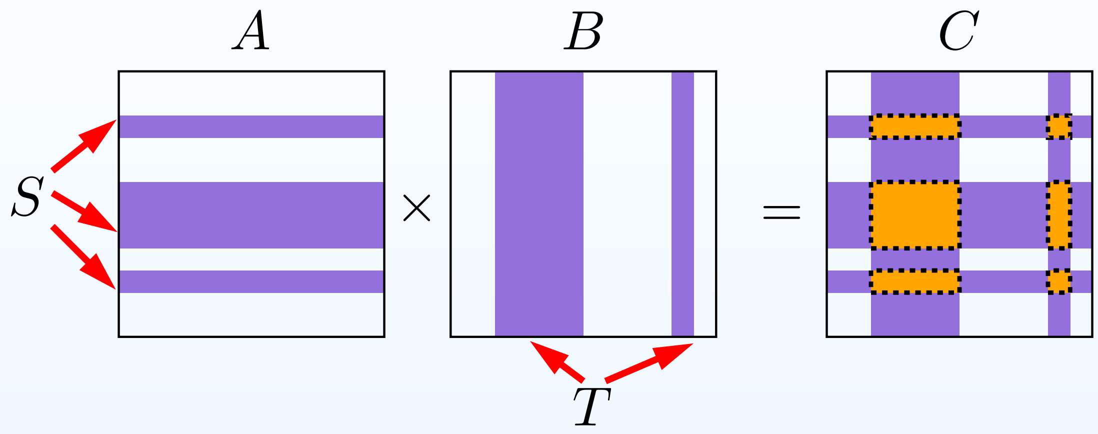

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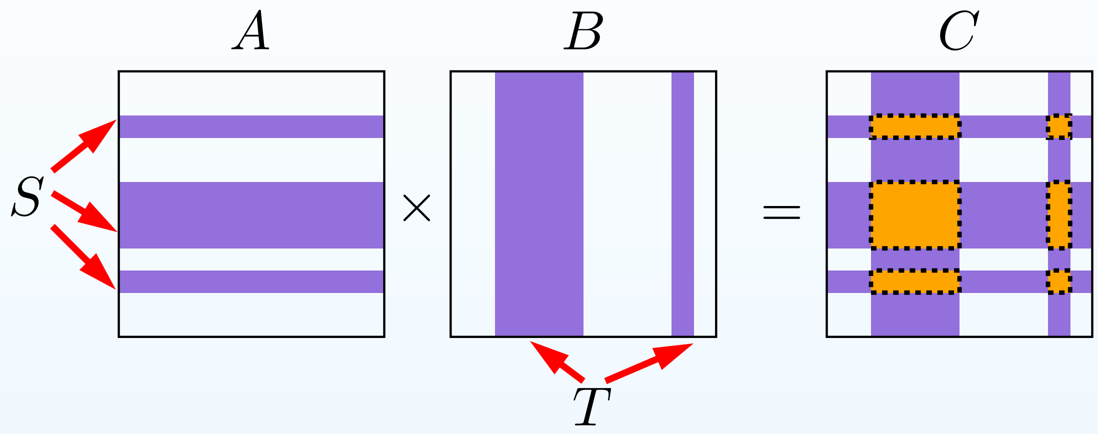

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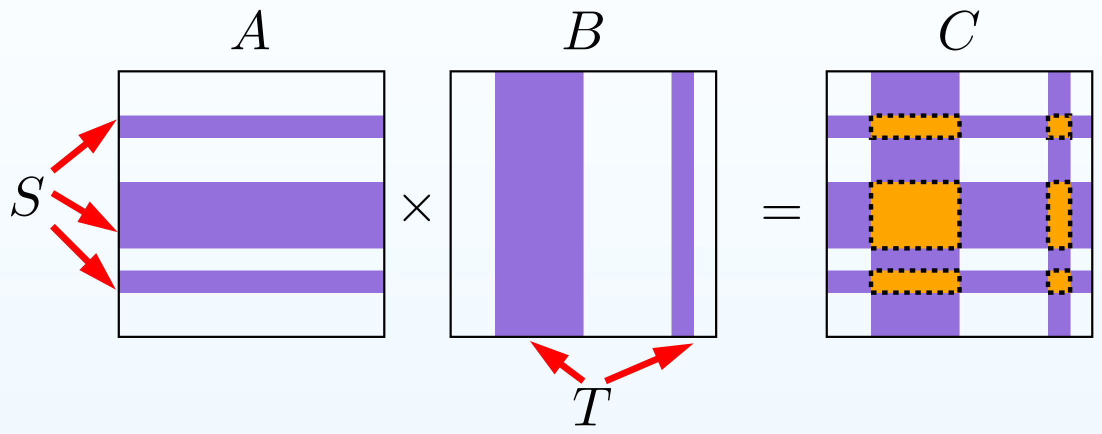

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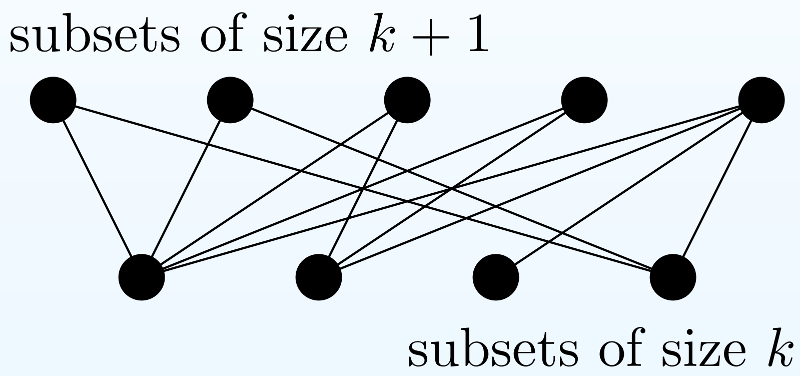
   (b) *Walk* with S, T by replacing one row and one column.
3. Measure S, T, and the submatrices,
   and verify classically the restricted matrix product.

Graph used in the algorithm

- *Johnson graph* $J(n,k)$ has vertices $\binom{[n]}{k} \cup \binom{[n]}{k+1}$ and edges between sets that differ in exactly one item



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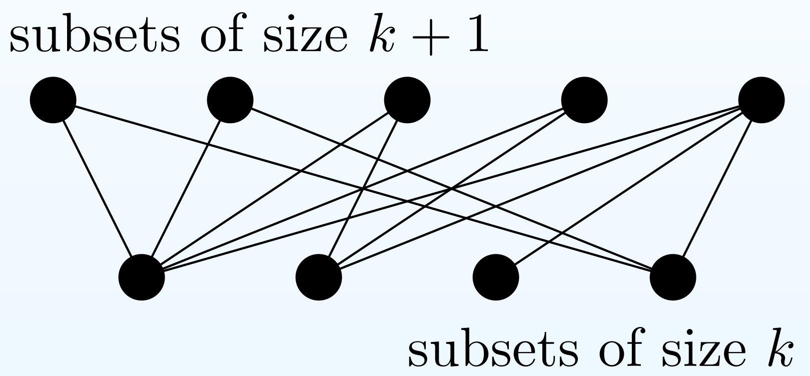

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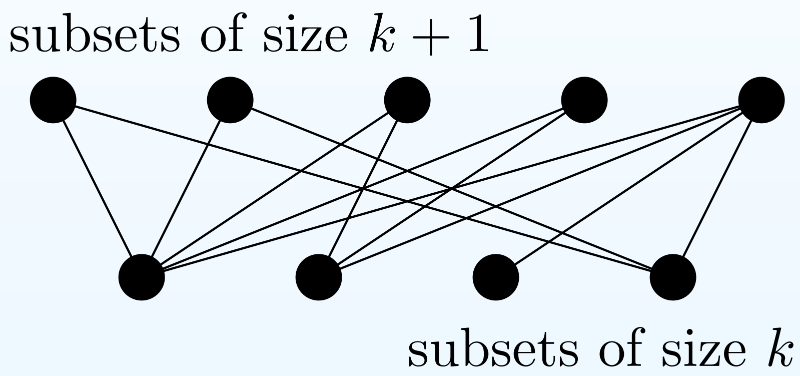

- The spectral gap of $J(n,k)$ is $\delta = \Theta(\frac{1}{k})$
- Our algorithm walks on the *strong product graph* $J(n,k) \times J(n,k)$, which has the same gap

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Q is minimal for $k = n^{2/3}$, and then it is $O(n^{5/3})$

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  2. space complexity

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- However, for $w > \sqrt{n}$, the speedup of verification is smaller, and the classical algorithm by [Indyk, 2005] with time $O(n^2 + nw)$ takes over.

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- Can we multiply dense matrices faster than classically?
- Matching lower bound?