Quantum Time-Space Tradeoffs for Deciding Systems of Linear Inequalities

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joint work with Andris Ambainis and Ronald de Wolf

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Time-Space Tradeoffs

- A relation between the running time and space complexity

The more memory is available,
the faster the algorithm can possibly run.

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- Example: sorting of N numbers

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  - Quantumly

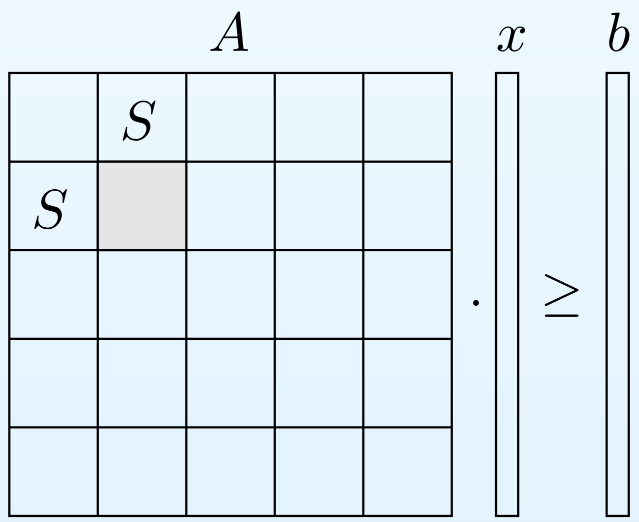
$$T^2 S = N^3 t, \quad S \leq N/t$$
$$TS = N^2, \quad S > N/t$$

    if numbers in b are *at most t*
- Omitting log-factors in the upper bounds

Upper Bound

Classical Algorithm

- Split the matrix into $(N/S)^2$ blocks of size $S \times S$



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  - maintain S counters at the same time
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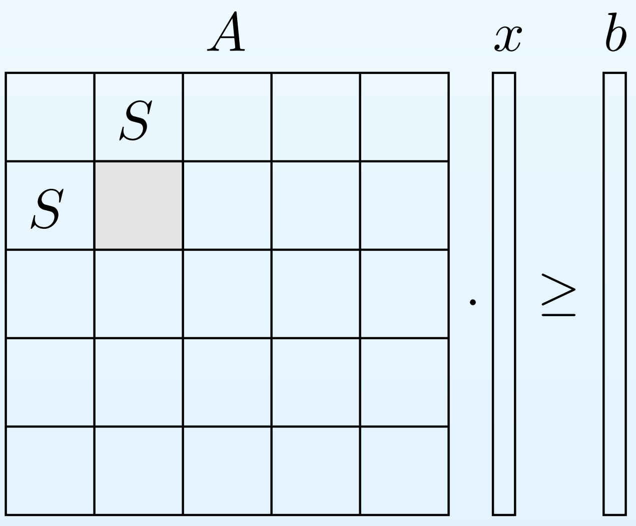

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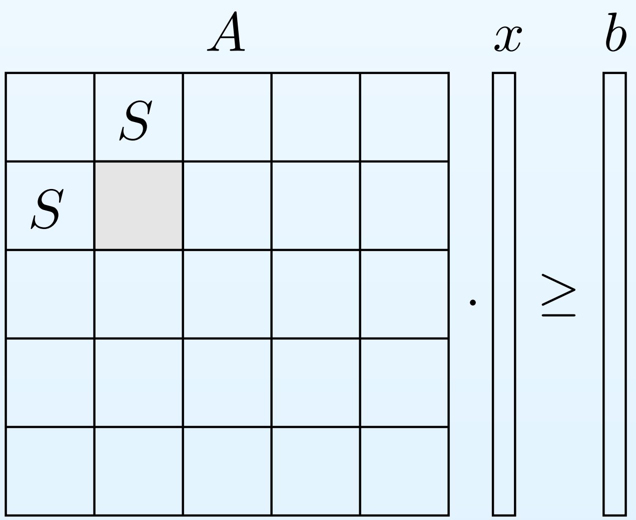

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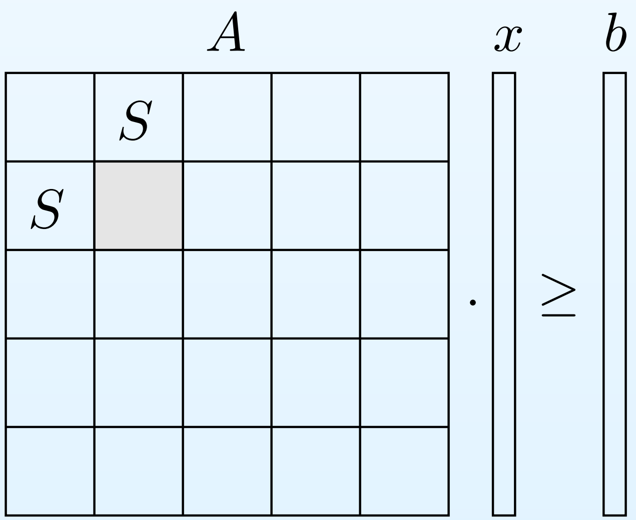
$TS \leq N^2$



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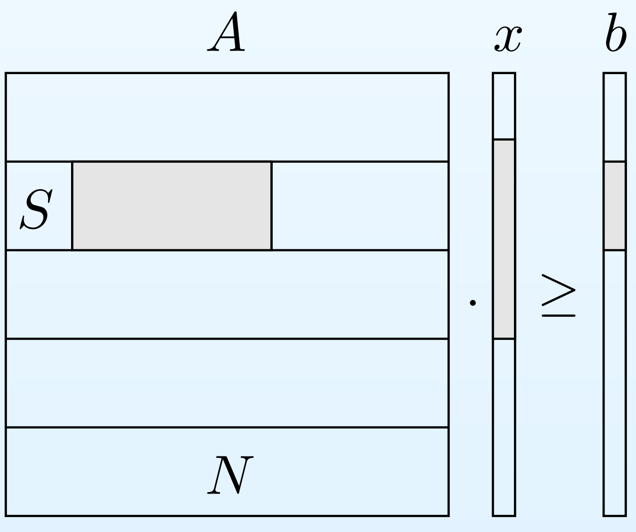

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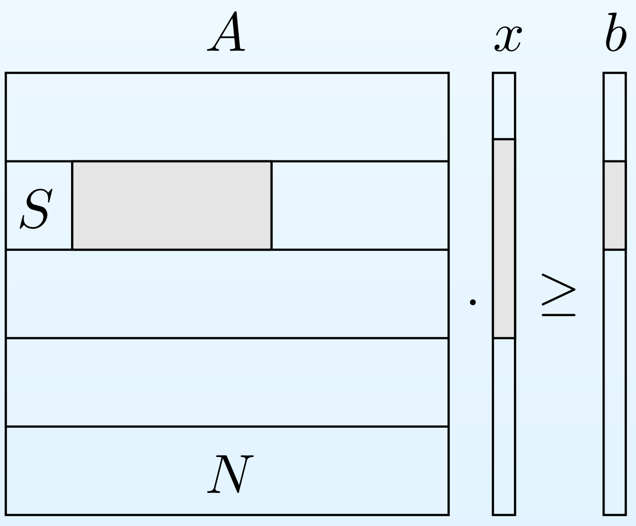

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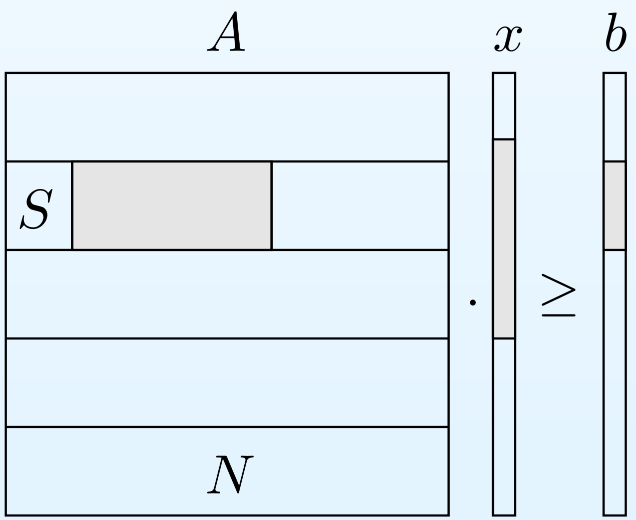

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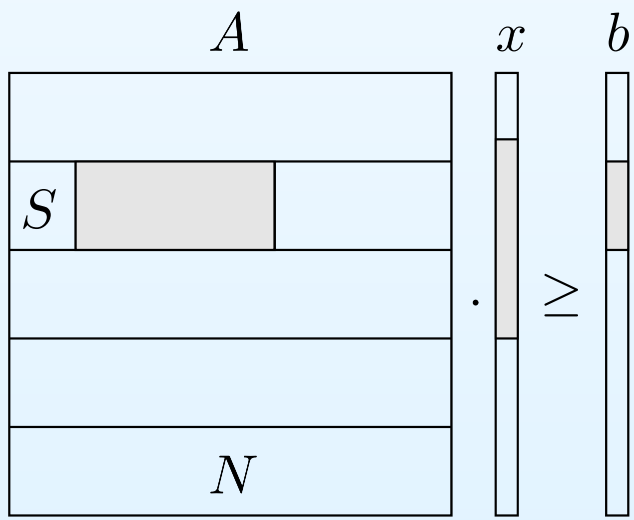
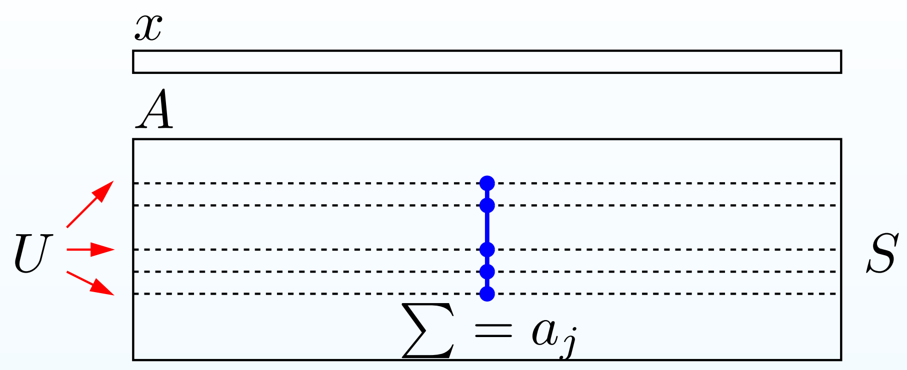


Quantum Algorithm (cont.)

y	vector of counters
U	set of *open rows* with $y_i < b_i$
a	column sum of A over the rows from U



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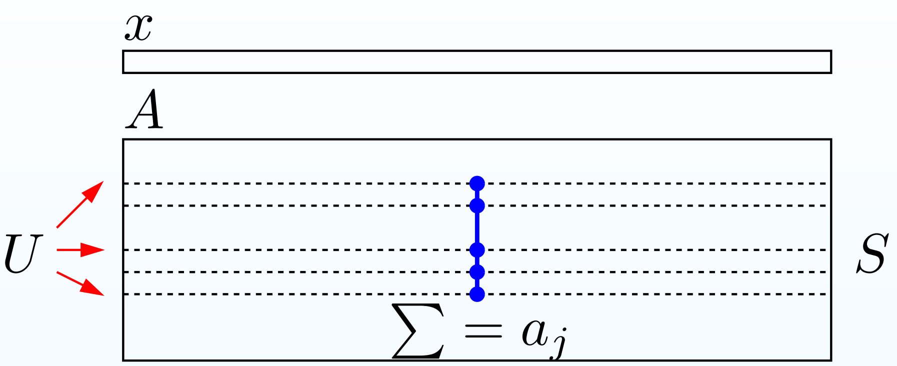

Start at position $p \leftarrow 1$ and with $U \leftarrow [1, S]$.

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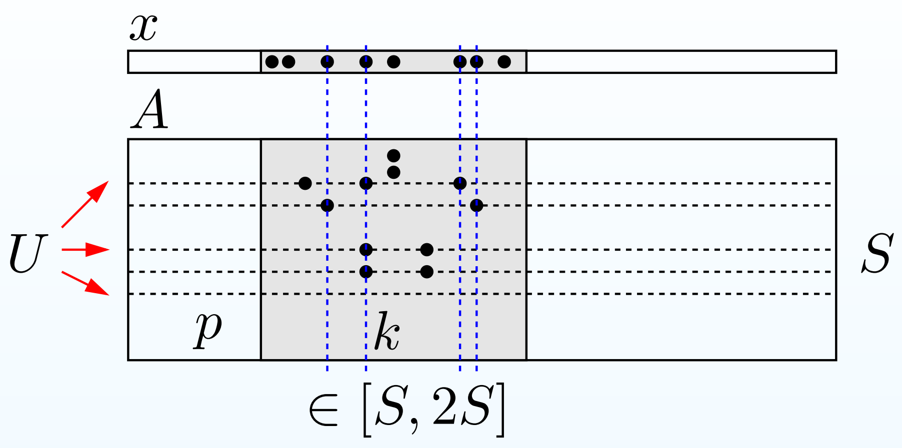

While $p \le N$ and $U \neq \emptyset$, do

- Find by binary search some k such that

$$S \le \sum_{j=p}^{p+k-1} a_j x_j \le 2S \qquad \text{. . . quantum counting}$$

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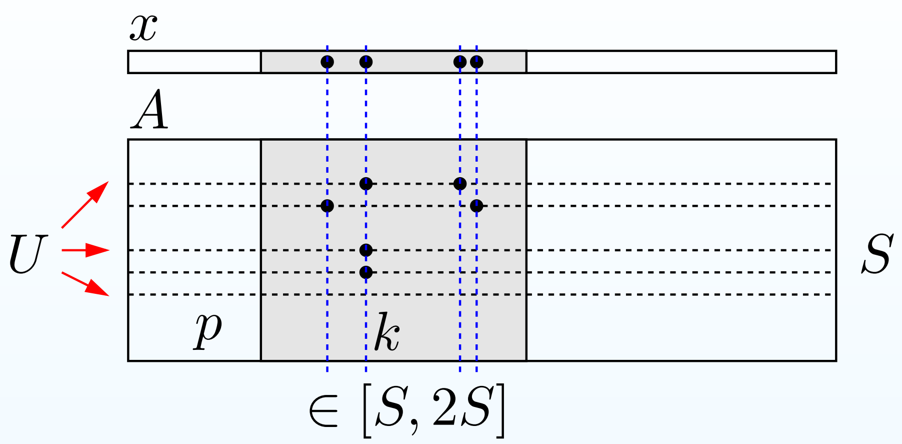

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$$a_j x_j > 0$$

...quantum search

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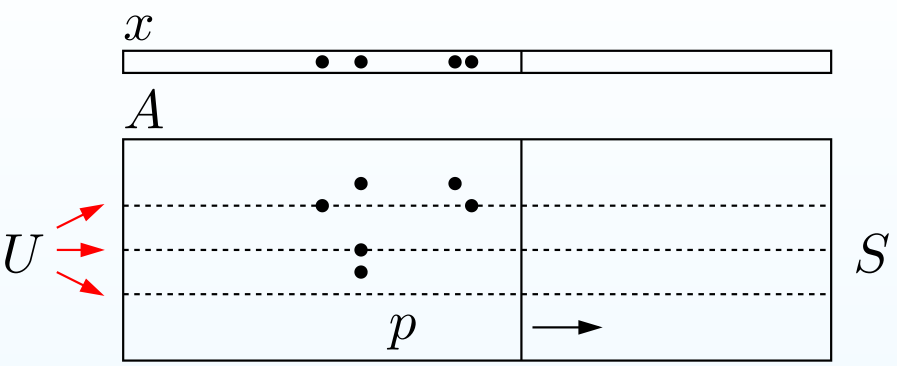

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- Find all positions j inside $[p, p+k-1]$ such that
$$a_j x_j > 0$$
...quantum search
- Update the counters y, remove from U the rows that have been closed in this iteration, and set $p \leftarrow p + k$

Complexity of the Algorithm

$i:$	1	2	3	4	5

In the i-th iteration of length k_i,

- cost of quantum counting with $\sqrt{k_i}$ queries is negligible
- quantum search costs $\sqrt{k_i r_i t} + \sqrt{k_i s_i}$, where
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By Cauchy-Schwarz,

$$
\begin{aligned}
T &= \sum_i \left(\sqrt{k_i r_i t} + \sqrt{k_i s_i} \right) \\
&\leq \sqrt{\sum k_i} \sqrt{t \sum r_i} + \sqrt{\sum k_i} \sqrt{\sum s_i} \\
&\leq \sqrt{NSt}
\end{aligned}
$$

Lower Bound

Direct Product Theorems

- Suppose we need $T(f)$ queries to compute f with small error. How hard is it to compute k independent instances $f(x_1), \ldots, f(x_k)$?

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- It is not known, whether the DPT holds in general!

  [Shaltiel, 2001]
  Counterexample for average-case complexity.
  However, DPT plausible for worst-case complexity.

Symmetric Functions

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- *Implicit threshold* is the minimal t such that f is constant on $[t, n-t]$. Example:
  - OR and AND have $t = 1$
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  - a-threshold function with $a \leq \frac{n}{2}$ has $t = a$
- Bounded-error quantum query complexity of a symmetric function is

$$Q_2(f) = \Theta(\sqrt{tn})$$

Quantum Query DPT

- [Klauck, Š, de Wolf, FOCS 2004]
  DPT for k instances of the OR function

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- [Ambainis, Š, de Wolf, 2005]
  Generalize [KŠW, Amb] to all symmetric functions f.

  $$T \leq \alpha k \sqrt{tn} \Rightarrow \sigma \leq 2^{-\gamma k}$$

  t is the implicit threshold of f

Constructing a Hard Matrix

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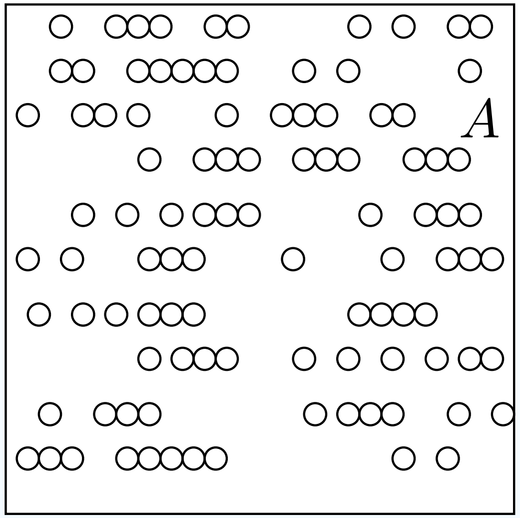

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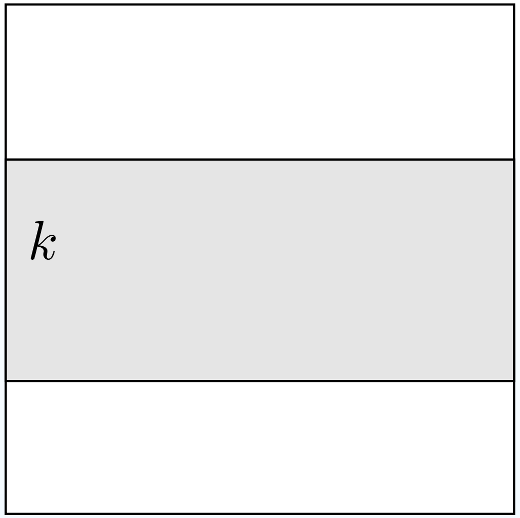

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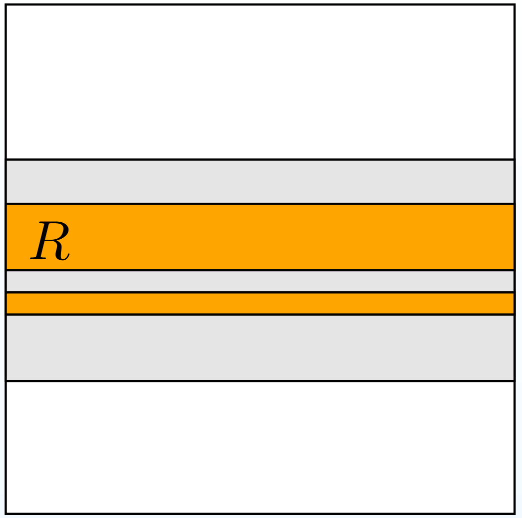

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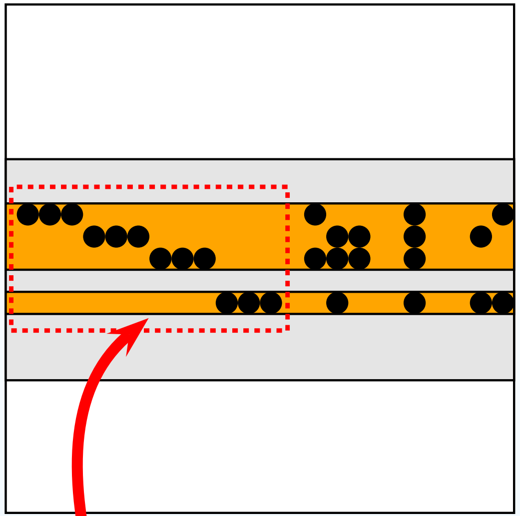

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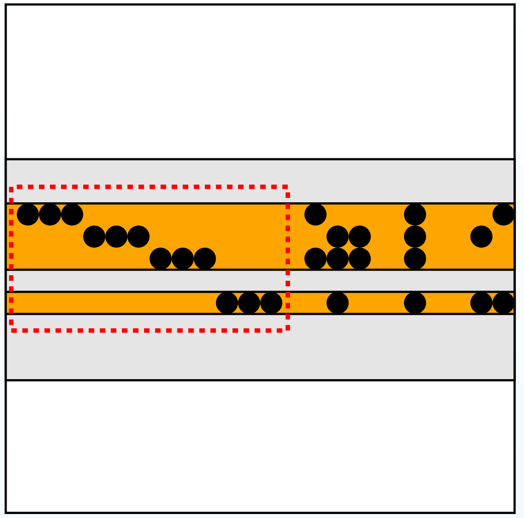

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Proof: pick $N/2k$ ones at random in each row.

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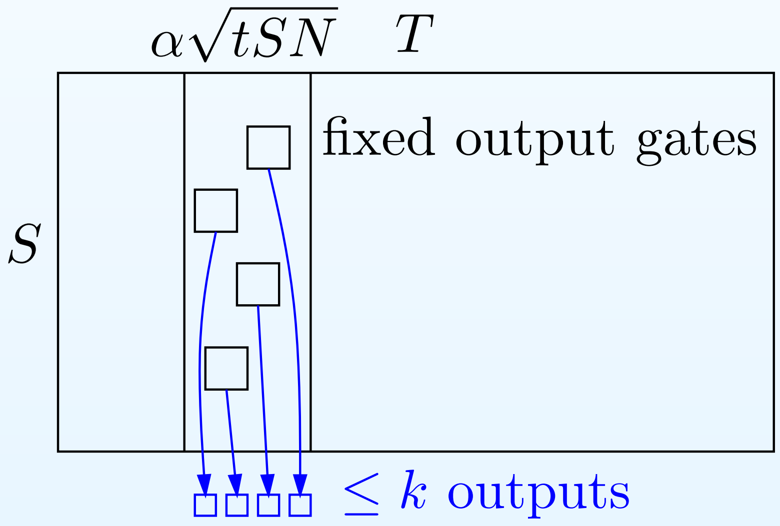

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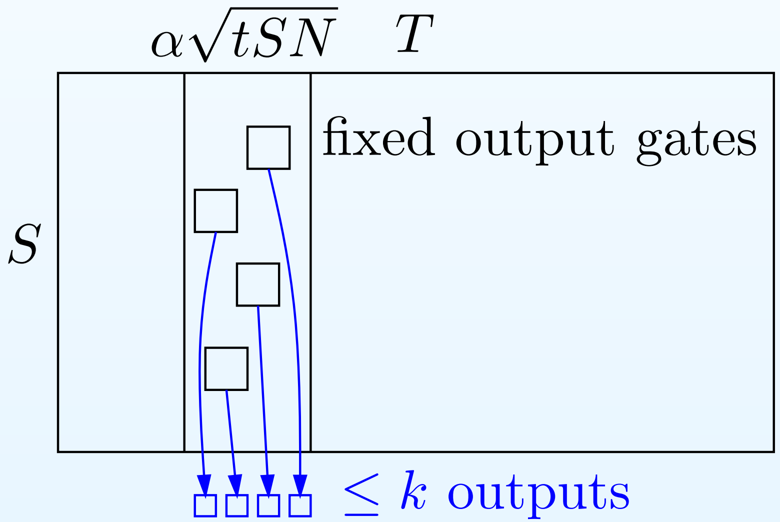

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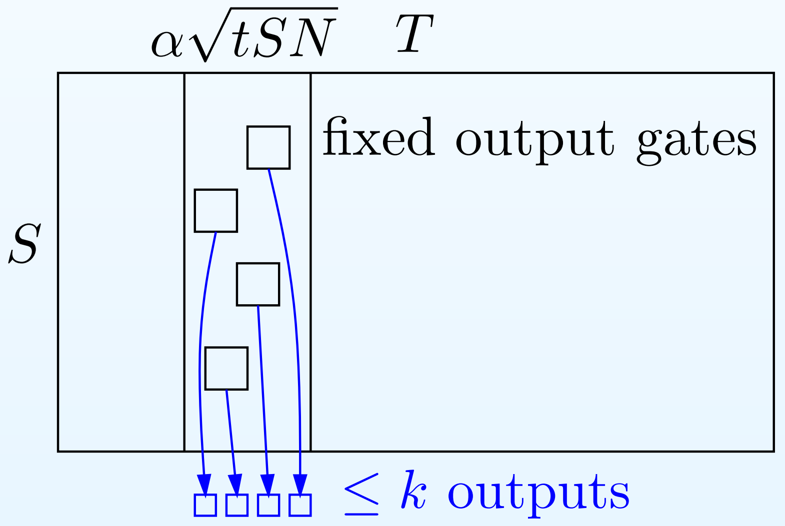

- We show that $k = O(S)$ due to the DPT
- $N \leq$ # slices $\cdot\, k = O\left(\frac{T\sqrt{S}}{\alpha\sqrt{tN}}\right)$, hence $T^2 S = \Omega(N^3 t)$

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- We do not have a matching upper bound, and we conjecture that the lower bound is not tight

Conclusion

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- A matching lower bound proved using direct product theorems
- For one-sided error algorithms the lower bound is stronger