Quantum Algorithms for Matching and Network Flows

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Abstract. We present quantum algorithms for some graph problems: finding a maximal bipartite matching in time $O(n\sqrt{m}\log n)$, finding a maximal non-bipartite matching in time $O(n^2(\sqrt{m/n}+\log n)\log n)$, and finding a maximal flow in an integer network in time $O(\min(n^{7/6}\sqrt{m}\cdot U^{1/3}, \sqrt{nU}m)\log n)$, where n is the number of vertices, m is the number of edges, and $U \leq n^{1/4}$ is an upper bound on the capacity of an edge.

1 Introduction

Network flows is one of the most studied problems in computer science. We are given a directed graph with two designated vertices: a source and a sink. Each edge has assigned a capacity. A network flow is an assignment of flows to the edges such that the capacity of an edge is never exceeded and the total incoming and outgoing flow are equal for each vertex except for the source and the sink. A size of the flow is the total flow going from the source. The task is to find a flow of maximal size.

After the pioneering work of Ford and Fulkerson [1], many algorithms have been proposed. Let n denote the number of vertices and let m denote the number of edges. For networks with real capacities, the fastest algorithms run in time $O(n^3)$ [2, 3]. If the network is sparse, one can achieve a faster time $O(nm(\log n)^2)$ [4]. If all capacities are integers bounded by U, the maximal flow can be found in time $O(\min(n^{2/3}m, m^{3/2})\log(n^2/m)\log U)$ [5]. For unit networks, the log-factor is not necessary and the fastest algorithm runs in time $O(\min(n^{2/3}m, m^{3/2}))$ [6]. For undirected unit networks, the fastest known deterministic algorithm runs in time $O(n^{7/6}m^{2/3})$ and the fastest known probabilistic algorithm runs in time $O(n^{20/9})$ [7].

Another well studied problem is finding a matching in a graph. We are given an undirected graph. A matching is a set of edges such that every vertex is connected to at most one other vertex. The task is to find a matching of maximal size. The simplest classical algorithm based on augmenting paths runs in time $O(n^3)$ [8, 9]. If the graph is bipartite, then the simple algorithm finds a maximal

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matching in faster time $O(n^{5/2})$ [10]. Finding a bipartite matching can be reduced to finding a maximal flow in a directed unit network, hence one can apply the same algorithms and achieve a running time $O(\min(n^{2/3}m, m^{3/2}))$ [6]. The fastest known algorithm for general sparse graphs runs in time $O(\sqrt{n}m)$ [11]. Recently, Mucha and Sankowski published a new algorithm [12] based on matrix multiplication that finds a maximal matching in general graphs in time $O(n^{\omega})$, where $2 \leq \omega \leq 2.38$ is the exponent of the best matrix multiplication algorithm.

In our paper, we analyze the quantum time complexity of these problems. We use Grover's search [13, 14] to speed up searching for an edge. A similar approach has been successfully applied by Dürr et al. [15] to the following graph problems: connectivity, strong connectivity, minimum spanning tree, and single source shortest paths. Our bipartite matching algorithm is polynomially faster than the best classical algorithm when $m = \Omega(n^{1+\varepsilon})$ for some $\varepsilon > 0$, and the network flows algorithm is polynomially faster when $m = \Omega(n^{1+\varepsilon})$ and U is small. Out non-bipartite matching algorithm is worse than the best known classical algorithm [11].

There is an $\Omega(n^{3/2})$ quantum adversary lower bound for the bipartite matching problem [16, 17]. Since the bipartite matching problem is a special case of the other problems studied in this paper, this implies an $\Omega(n^{3/2})$ quantum lower bound for all problems in this paper.

2 Preliminaries

An excellent book about quantum computing is the textbook by Nielsen and Chuang [18]. In this paper, we only use two quantum sub-routines and otherwise our algorithm are completely classical. The first one is a generalization of Grover's search that finds k items in a search space of size ℓ in total time $O(\sqrt{k\ell})$ [13, 14]. An additional time $O(\sqrt{\ell})$ is needed to detect that there are no more solutions; this term is only important when $k = 0$. The second one is quantum counting that estimates the number of ones in a string of length n within additive constant $\sqrt{n}$ with high probability in time $O(\sqrt{n})$ [19, Theorem 13].

Each of those algorithms may output an incorrect answer with a constant probability. Our algorithms may use a polynomial number n^c of quantum subroutines. Because of that, we have to repeat each quantum subroutine $O(\log n)$ times, to make sure that the probability of an incorrect answer is less than $1/n^{c+1}$. Then, the probability that all quantum subroutines in our algorithm output the correct answer is at least $1 - 1/n$. This increases the running time of all our algorithms by a $\log n$ factor. We omit the log-factors in the proofs, but we state them in the statements of our theorems.

A very good book about network flows is the classical book by Ahuja, Magnanti, and Orlin [20]. It, however, does not contain most of the newest algorithms that we compare our algorithms to. We use the following concepts: A *layered network* is a network whose vertices are ordered into a number of layers, and whose edges only go from the i-th layer to the $(i+1)$-th layer. A *residual network* is a network whose capacities denote the residual capacity of the edges in the original network. When an edge has a capacity c and carries a flow f, then its residual capacity is either $c - f$ or $c + f$ depending on the direction. An *augmenting path* in a network is a path from the source to the sink whose residual capacity is bigger than 0. An augmenting path for the matching problem is a path that consists of alternated non-edges and edges of the current matching, and starts and ends in a free vertex. A *blocking flow* in a layered residual network is a maximal flow with respect to inclusion. A blocking flow cannot be increased by one augmenting path. A *cut* in a network is a subset of edges such that there is no path from the source to the sink if we remove these edges. The size of a cut is the sum of the capacities of its edges. Any flow has size smaller or equal to the size of any cut.

Let us define our computational model. Let V be a fixed vertex set of size $n \geq 1$ and let $E \subseteq \binom{V}{2}$ be a set of edges. E is a part of the input. Let m denote the number of edges. We assume that $m \geq n$, since one can eliminate zero-degree vertices in classical time $O(n)$. We consider the following two black-box models for accessing directed graphs:

- *Adjacency* model: the input is specified by an $n \times n$ Boolean matrix A, where $A[v, w] = 1$ iff $(v, w) \in E$.
- *List* model: the input is specified by n arrays $\{N_v : v \in V\}$ of length $1 \leq d_v \leq n$. Each entry of an array is either a number of a neighbor or an hole, and $\{N_v[i] : i = 1, \ldots, d_v\} - \{\text{hole}\} = \{w : (v, w) \in E\}$.

The structure of the paper is as follows: In Section 3, we present a quantum algorithm for computing a layered network from a given network. It is used as a tool in almost all our algorithms. In Section 4, we present a simple quantum algorithm for bipartite matching. In Section 5, we show how to quantize the classical algorithm for non-bipartite matching. In Section 6, we present a quantum algorithm for network flows.

3 Finding a layered subgraph

We are given a connected directed black-box graph $G = (V, E)$ and a starting vertex $a \in V$, and we want to assign layers $\ell : V \to \mathbb{N}$ to its vertices such that $\ell(a) = 0$ and $\ell(y) = 1 + \min_{x:(x,y) \in E} \ell(x)$ otherwise. The following quantum algorithm computes layer numbers for all vertices:

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1. Set $\ell(a) = 0$ and $\ell(x) = \infty$ for $x \neq a$.
   Create a one-entry queue $W = \{a\}$.
2. While $W \neq \emptyset$,
     – take the first vertex $x$ from $W$,
     – find by Grover's search all its neighbors $y$ with $\ell(y) = \infty$,
       set $\ell(y) := \ell(x) + 1$, and append $y$ into $W$,
     – and remove $x$ from $W$.
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Theorem 1. *The algorithm assigns layers in time $O(n^{3/2}\log n)$ in the adjacency model and in time $O(\sqrt{nm}\log n)$ in the list model.*

Proof. The algorithm is a quantum implementation of breadth-first search. The initialization costs time $O(n)$. Every vertex is processed at most once. In the adjacency model, every vertex contributes by time at most $O(\sqrt{n})$, because finding a vertex from its ancestor costs time at most $O(\sqrt{n})$ and discovering that a vertex has no descendant costs the same.

In the list model, processing a vertex v costs time $O(\sqrt{n_v d_v} + \sqrt{d_v + 1})$, where n_v is the number of vertices inserted into W when processing v. Let $f \leq \min(n, m)$ be the number of found vertices. Since $\sum_v n_v \leq f \leq n$ and $\sum_v (d_v + 1) \leq m + f = O(m)$, the total running time is upper-bounded by the Cauchy-Schwarz inequality as follows:

$$\sum_v \sqrt{n_v d_v} \leq \sqrt{\sum_v n_v}\sqrt{\sum_v d_v} = O(\sqrt{nm}),$$

and $\sum_v \sqrt{d_v + 1} \leq \sqrt{f}\sqrt{m + f}$ is upper-bounded in the same way.

4 Bipartite matching

We are given an undirected bipartite black-box graph $G = (V_1, V_2, E)$ and we want to find a maximum matching among its vertices. This can be done classically in time $O(n^{5/2})$ [10] as follows:

1. Set M to an empty matching.
2. Let $H = (V', E')$ denote the following graph:

$$\begin{aligned} V' &= V_1 \cup V_2 \cup \{a, b\} \\ E' &= \{(a, x) : x \in V_1, x \notin M\} \\ &\quad \cup \{(x, y) : x \in V_1, y \in V_2, (x, y) \in E, (x, y) \notin M\} \\ &\quad \cup \{(y, x) : x \in V_1, y \in V_2, (x, y) \in E, (x, y) \in M\} \\ &\quad \cup \{(y, b) : y \in V_2, y \notin M\}, \end{aligned}$$

   where the shortcut $x \notin M$ means that x is not matched.

   Find a maximal (with respect to inclusion) set S of vertex-disjoint augmenting paths of minimal length. This is done as follows: First, construct a layered subgraph H' of H. Second, perform a depth-first search for a maximal set of vertex-disjoint paths from a to b in H'. Every such a path is an augmenting path in M, and they all have the same minimal length.
3. Augment the matching M by S.
4. If $S \neq \emptyset$, go back to step 2, otherwise output the matching M.

The algorithm is correct because (1) a matching is maximal iff there is no augmenting path, and (2) the minimal length of an augmenting path is increased by at least one after every iteration. The construction of H' classically and the depth-first search both cost $O(n^2)$. The maximal number of iterations is $O(\sqrt{n})$ due to the following statement:

Lemma 1. [10] *If M_1 and M_2 are two matchings of size s_1 and s_2 with $s_1 < s_2$, then there exist $s_2 - s_1$ vertex-disjoint augmenting paths in M_1.*

Let s be the size of the maximal matching M in G, and let s_i be the size of the found matching M_i after the i-th iteration. Let j be the number of the first iteration with $s_j \geq s - \sqrt{n}$. The total number of iterations is at most $j + \sqrt{n}$, because the algorithm finds at least one augmenting path in every iteration. On the other hand, by Lemma 1, there are $s - s_j \geq \sqrt{n}$ vertex-disjoint augmenting paths in M_j. Since all augmenting paths in the j-th iteration are of length at least $j + 2$, it must be that $j < \sqrt{n}$, otherwise the paths would not be disjoint. We conclude that the total number of iterations is at most $2\sqrt{n}$.

Theorem 2. *Quantumly, a maximal bipartite matching can be found in time $O(n^2 \log n)$ in the adjacency model and $O(n\sqrt{m} \log n)$ in the list model.*

Proof. We present a quantum algorithm that finds all augmenting paths in one iteration in time $O(n^{3/2})$, resp. $O(\sqrt{nm})$, times a log-factor for Grover's search. Since the number of iterations is $O(\sqrt{n})$, the upper bound on the running time follows. Our algorithm works similarly to the classical one; it also computes the layered graph H' and then searches in it.

The intermediate graph H is generated on-line from the input black-box graph G and the current matching M, using a constant number of queries as follows: the sub-graph of H on $V_1 \times V_2$ is the same as G except that some edges have been removed; here we exploit the fact that the lists of neighbors can contain holes. We also add two new vertices a and b, add one list of neighbors of a with holes of total length n, and at most one neighbor b to every vertex from V_2. Theorem 1 states how long it takes to compute H' from H. It remains to show how to find the augmenting paths in the same time.

This is simple once we have computed the layer numbers of all vertices. We find a maximal set of vertex-disjoint paths from a to b by a depth-first search. A descendant of a vertex is found by Grover's search over all unmarked vertices with layer number by one bigger. All vertices are unmarked in the beginning. When we find a descendant, we mark it and continue backtracking. Either the vertex will become a part of an augmenting path, or it does not belong to any and hence it needs not be tried again. Each vertex is thus visited at most once.

In the adjacency model, every vertex costs time $O(\sqrt{n})$ to be found and time $O(\sqrt{n})$ to discover that it does not have any descendant. In the list model, a vertex v costs time $O(\sqrt{n_v d_v} + \sqrt{d_v})$, where n_v is the number of unmarked vertices found from v. The sum over all vertices is upper-bounded like in the proof of Theorem 1. Note that $\sum_v d_v$ has been increased by at most $2n$.

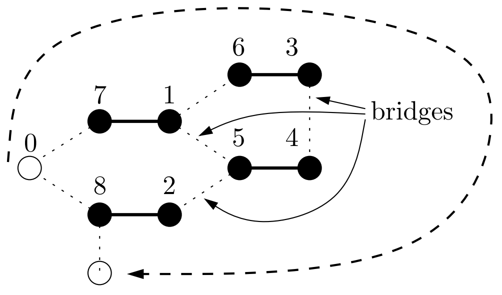
5 Non-bipartite matching

We are given an undirected graph $G = (V, E)$ and we want to find a maximal matching among its vertices. There is a classical algorithm [8, 9] running in total time $O(n^3)$ in n iterations of time $O(n^2)$.

Each iteration consists of searching for an augmenting path. The algorithm performs a breadth-first search from some free vertex. It browses paths that consist of alternated non-edges and edges of the current matching. The matching is specified by pointers *mate*. Let us call a vertex v *even* if we have found such an alternated path of even length from the start to v; otherwise we call it *odd*. Newly discovered vertices are considered to be odd. For each even vertex, we store two pointers *link* and *bridge* used for tracing the path back, and a pointer *first* to the last odd vertex on this path. The algorithm works as follows and its progress on an example graph is outlined in Figure 1:

1. Initialize a queue of even vertices $W = \{a\}$ with some free vertex a.
2. Take the first vertex v from W and delete it from W.
3. If there exists an free vertex w connected to v, then augment the current matching by the path $a \to v$ plus the edge (v, w), and quit. A general subpath $\rho : b \to v$ is traced recursively using v's pointers as follows:
   - If *bridge* is nil, then *link* points to the previous even vertex on ρ. Output 2 edges from v to *mate* and *link*, and trace ρ from *link* to b.
   - Otherwise v was discovered via a bridge, *link* points to v's side of the bridge, and *bridge* to the other side. Trace ρ from *link* to v in the opposite direction, and then from *bridge* to b in the normal direction.
4. For every odd vertex w connected to v, do the following:
   - Let w be connected to a mate w'. If w' is even, do nothing.
   - Otherwise mark w' as even, append it to W, and set its pointers as follows: *link* to v, *bridge* to nil, and *first* to w.
5. For every even vertex w connected to v, do the following:
   - Compare the pointers *first* of v and w. If they are equal, do nothing.
   - Now, v and w lie on a circle of odd length, and the edge (v, w) is a *bridge* between the two subpaths. Find the nearest common odd ancestor p of v and w by tracing the pointers *first*. Collapse the circle as follows:
     - Mark all odd vertices between v and p as even, append them to W, and set their pointers as follows: *link* to v, *bridge* to w, and *first* to p.
     - Do the same for odd vertices between w and p.
     - Finally, rewrite all links *first* pointing to odd vertices that have just become even to p.
6. If W is empty, then there is no augmenting path from a and we quit, otherwise go back to step 2.

It holds that if an augmenting path from some vertex has not been found, then it would not be found even later after more iterations of the algorithm. Hence it suffices to search for an augmenting path from each vertex once.



- white vertices are free
- thick solid lines denote the current matching
- dotted lines are the remaining edges of the graph
- even vertices are numbered in the order they are found
- vertices 5, 6, 7, and 8 are found via 3 bridges
- the augmenting path is 0, 7, 1, 6, 3, 4, 5, 2, 8, and the final free vertex

Fig. 1. The classical non-bipartite matching algorithm [8, 9]

Theorem 3. *Quantumly, a maximal non-bipartite matching can be found in time* $O(n^{5/2} \log n)$ *in the adjacency model and* $O(n^2(\sqrt{m/n} + \log n) \log n)$ *in the list model.*

Proof. The algorithm iteratively augments the current matching by single augmenting paths, like the classical algorithm. An augmenting path is found using Grover's search in faster time $O(n^{3/2})$, resp. $O(n(\sqrt{m/n} + \log n))$, times the usual log-factor. This implies the bound on the total running time, since there are n vertices and each of them is used as the starting vertex a at most once. Let us prove the time bound for the list model.

Let $f \leq \min(n, m)$ denote the number of even vertices. For every even vertex v, we perform the following 3 Grover's searches: First, we look for a free neighbor of v in time $O(\sqrt{d_v})$. Second, we process all odd neighbors of v whose mate is still odd in total time $O(\sqrt{e_v d_v})$, where e_v is the number of odd vertices that are found during processing v. Third, we process all even neighbors of v whose pointer *first* is different from v's pointer *first*, in time $O(\sqrt{b_v d_v})$, where b_v is the number of bridges that are found during processing v. Clearly $\sum_v e_v \leq f$ and $\sum_v b_v \leq f$, and, since $\sum_v d_v \leq m$, by the Cauchy-Schwarz inequality, the total time spent in all Grover's searches is $O(\sqrt{nm})$.

Let us estimate the running time of collapsing one circle. Let p_1 be the length of the link-list of pointers *first* from one side of the bridge into the nearest common parent, let p_2 be the other one, and let $p = \max(p_1, p_2)$. The nearest common parent is found in time $O(p \log p)$ as follows: we maintain two balanced binary trees for each link-list, add vertices synchronously one-by-one, and search for every newly inserted vertex in the opposite tree, until we find a collision. Let r_v be the number of odd vertices collapsed during processing a vertex v. It holds that $r_v = p_1 + p_2 = \Theta(p)$ and $\sum_v r_v \leq f$. Hence the total time spent in collapsing circles is $O(f \log f)$.

Rewriting the pointers *first* of all even vertices inside a collapsed circle would be too slow. We instead maintain aside a Union-tree of all these pointers, and for every odd vertex converted to even, we append its subtree to the node of the nearest common ancestor. The total time spent in doing this is $O(f \log f)$.

The augmenting path has length at most n and it is traced back in linear time. We conclude that the total running time of finding an augmented time is $O(\sqrt{nm} + n \log n) = O(n(\sqrt{m/n} + \log n))$, which is $O(\sqrt{nm})$ for $m \geq n(\log n)^2$. The running time in the adjacency model is equal to the running time in the list model with $m = n^2$, that is $O(n^{5/2})$.

It would be interesting to quantize the fastest known classical algorithm by Micali and Vazirani [11] running in total time $O(\sqrt{n}m)$.

6 Integer network flows

We are given a directed network with real capacities, and we want to find a maximal flow from the source to the sink. There are classical algorithms running in time $O(n^3)$ [2,3]. They iteratively augment the current flow by adding blocking flows in layered residual networks [21] of increasing depth. Since the depth is increased by at least one after each iteration, there are at most n iterations. Each of them can be processed in time $O(n^2)$. For sparse real networks, the fastest known algorithm runs in time $O(nm(\log n)^2)$ [4].

Let us restrict the setting to integer capacities bounded by U. There is a simple capacity scaling algorithm running in time $O(nm \log U)$ [22,21]. The fastest known algorithm runs in time $O(\min(n^{2/3}m, m^{3/2}) \log(n^2/m) \log U)$ [5]. For unit networks, i.e. for $U = 1$, a simple combination of the capacity scaling algorithm and the blocking-flow algorithm runs in time $O(\min(n^{2/3}m, m^{3/2}))$ [6]. For undirected unit networks, there is an algorithm running in time $O(n^{3/2}\sqrt{m})$ [23], and the fastest known algorithm runs in worst-case time $O(n^{7/6}m^{2/3})$ and expected time $O(n^{20/9})$ [7].

Lemma 2. [6] *Let us have an integer network with capacities bounded by U, whose layered residual network has depth k. Then the size of the residual flow is at most* $\min((2n/k)^2, m/k) \cdot U$.

Proof. (1) There exist layers V_ℓ and $V_{\ell+1}$ that both have less than $2n/k$ vertices. This is because if for every $i = 0, 1, \ldots, k/2$, at least one of the layers V_{2i}, V_{2i+1} had size at least $2n/k$, then the total number of vertices would exceed n. Since V_ℓ and $V_{\ell+1}$ form a cut, the residual flow has size at most $|V_\ell| \cdot |V_{\ell+1}| \cdot U \leq (2n/k)^2 U$.

(2) For every $i = 0, 1, \ldots, k-1$, the layers V_i and V_{i+1} form a cut. These cuts are disjoint and they together have at most m edges. Hence at least one of them has at most m/k edges, and the residual flow has thus size at most $O(mU/k)$.

Theorem 4. *Let $U \leq n^{1/4}$. Quantumly, a maximal network flow with integer capacities at most U can be found in time $O(n^{13/6} \cdot U^{1/3} \log n)$ in the adjacency model and in time $O(\min(n^{7/6}\sqrt{m} \cdot U^{1/3}, \sqrt{nU}m) \log n)$ in the list model.*

Proof. The algorithm iteratively augments the current flow by blocking flows in layered residual networks [21], until the depth of the network exceeds $k = \min(n^{2/3}U^{1/3}, \sqrt{mU})$. Then it switches to searching augmenting paths [22], while there are some. The idea of switching the two algorithms comes from [6]. Our algorithm uses classical memory of size $O(n^2)$ to store the current flow and its direction for every edge of the network, and a 1-bit status of each vertex. A blocking flow is found as follows:

1. Compute a layered subgraph H' of the *residual* network H. The capacity of each edge in H is equal to the original capacity plus or minus the current flow depending on the direction. Edges with zero capacities are omitted.
2. Mark all vertices as enabled.
3. Find by a depth-first search a path ρ in H' from the source to the sink that only goes through enabled vertices. If there is no such a path, quit. During back-tracking, disable all vertices from which there is no path to the sink.
4. Compute the minimal capacity μ of an edge on ρ.
   Augment the flow by μ along ρ.
5. Go back to step 3.

The layered subgraph H' is computed from H using Theorem 1, and the capacities of H are computed on-line in constant time. When the flow is augmented by μ along the path ρ, the saturated edges will have been automatically deleted. This is because the algorithm only stores layer numbers for the vertices, and the edges of H' are searched on-line by Grover's search.

Let us compute how much time the algorithm spends in a vertex v during searching the augmenting paths. Let a_v denote the number of augmenting paths going through v and let $e_{v,i}$ denote the number of outgoing edges from v at the moment when there are still i remaining augmenting paths. The capacity of every edge is at most U, hence $e_{v,i} \geq \lceil i/U \rceil$. The time spent in Grover's searches leading to an augmenting path in v is thus at most

$$\sum_{i=1}^{a_v} \sqrt{\frac{d_v}{e_{v,i}}} \leq \sqrt{U} \cdot \sum_{i=1}^{a_v} \sqrt{\frac{d_v}{i}} = O(\sqrt{U a_v d_v}).$$

Let c_v denote the number of enabled vertices found from v that do not lie on an augmenting path and are thus disabled. The time spent in Grover's searches for these vertices is at most $O(\sqrt{c_v d_v})$. Furthermore, it takes additional time $O(\sqrt{d_v + 1})$ to discover that there is no augmenting path from v, and in this case v is disabled and never visited again.

Let j denote the depth of the network and let A_j be the size of its blocking flow. The total number of augmenting paths going through vertices in any given layer is at most A_j. We conclude that $\sum_v a_v \leq jA_j$. We also know that $\sum_v c_v \leq n$. Since $\sum_v d_v \leq m$, by the Cauchy-Schwarz inequality, the total time spent by finding one blocking flow is

$$\sum_v (\sqrt{Ua_v d_v} + \sqrt{c_v d_v} + \sqrt{d_v + 1}) \leq \sqrt{U}\sqrt{\sum_v a_v}\sqrt{\sum_v d_v} + 2\sqrt{nm}$$
$$= O(\sqrt{jmA_jU} + \sqrt{nm}).$$

Our algorithm performs at most $k = \min(n^{2/3}U^{1/3}, \sqrt{mU})$ iterations of finding the blocking flow in total time at most $\sqrt{mU} \cdot \sum_{j=1}^{k} \sqrt{jA_j} + k\sqrt{nm}$. Let us assume that the algorithm has not finished, and estimate the size of the residual flow and thus upper-bound the number of augmenting paths that need to be found. The algorithm has constructed in this iteration a layered network of depth bigger than k. By Lemma 2, the residual flow has size $O(\min((n/k)^2, m/k) \cdot U) = O(k)$, hence the algorithm terminates in $O(k)$ more iterations. From this point on, the algorithm only looks for one augmenting path in each layered network, hence its complexity drops to $O(\sqrt{j'm}) = O(\sqrt{nm})$ per iteration, omitting the factor $\sqrt{A_{j'}U}$. The total running time is thus at most

$$O\Big(\sqrt{mU} \cdot \sum_{j=1}^{k} \sqrt{jA_j} + k\sqrt{nm}\Big) + O(k\sqrt{nm}).$$

Let us prove that $\sum_j \sqrt{jA_j} = O(k^{3/2})$. We split the sequence into $\log k$ intervals $S_i = \{2^i, 2^i + 1, \ldots, 2^{i+1} - 1\}$ of length 2^i. By Lemma 2, the residual flow after $\ell = k/2^i$ iterations is at most $O(\min((n/k)^2 \cdot 2^{2i}, m/k \cdot 2^i) \cdot U) \leq O(2^{2i}k) = O((k/\ell)^2\ell) = O(k^2/\ell)$. Since the total size of all blocking flows cannot exceed the residual flow, $\sum_{j=\ell}^{2\ell-1} A_j = O(k^2/\ell)$. By applying the Cauchy-Schwarz inequality independently on each block, we get

$$\sum_{j=1}^{k} \sqrt{jA_j} = \sum_{i=0}^{\log k} \sum_{j=2^i}^{2^{i+1}-1} \sqrt{jA_j} \leq \sum_{i=0}^{\log k} \sqrt{2^i \cdot 2^{i+1}} \sqrt{\sum_{j=2^i}^{2^{i+1}-1} A_j}$$
$$\leq \sqrt{2} \sum_{i=0}^{\log k} 2^i \sqrt{k^2/2^i} = \sqrt{2} \cdot k \sum_{k=0}^{\log k} 2^{i/2} = O(k^{3/2}).$$

The total running time is thus $O(k\sqrt{m}(\sqrt{kU} + \sqrt{n}))$. Now, $kU \leq n$, because $U \leq n^{1/4}$ and $kU = \min(n^{2/3}U^{4/3}, \sqrt{m} \cdot U^{3/2}) \leq n^{2/3}n^{1/3} = n$. The running time is therefore $O(k\sqrt{nm}) = O(\min(n^{7/6}\sqrt{m} \cdot U^{1/3}, \sqrt{nU}m))$, times a log-factor for Grover's search. The time for the adjacency model follows from setting $m = n^2$ and it is $O(n^{13/6} \cdot U^{1/3} \log n)$.

It is not hard to compute an upper bound on the running time of the network flows algorithm for $U > n^{1/4}$ by the same techniques. One obtains

$O(\min(n^{7/6}\sqrt{m}, \sqrt{n}m) \cdot U \log n)$ for arbitrary U by setting $k = \min(n^{2/3}, \sqrt{m})$. It would be interesting to apply techniques of [5] to improve the multiplicative constant in Theorem 4 from $\mathrm{poly}(U)$ to $\log U$. If $m = \Omega(n^{1+\varepsilon})$ for some $\varepsilon > 0$ and U is small, then our algorithm is polynomially faster than the best classical algorithm. For constant U and $m = O(n)$, it is slower by at most a log-factor. The speedup is biggest for dense networks with $m = \Omega(n^2)$.

Theorem 5. *Any bounded-error quantum algorithm for network flows with integer capacities bounded by $U = n$ has quantum query complexity $\Omega(n^2)$.*

Proof. Consider the following layered graph with $m = \Theta(n^2)$ edges. The vertices are ordered into 4 layers: the first layer contains the source, the second and third layer contain $p = \frac{n}{2} - 1$ vertices each, and the last layer contains the sink. The source and the sink are both connected to all vertices in the neighboring layer by p edges of full capacity n. The vertices in the second and third layer are connected by either $\frac{p^2}{2}$ or $\frac{p^2}{2} + 1$ edges of capacity 1 chosen at random. The edges between these two layers form a minimal cut. Now, deciding whether the maximal flow is $\frac{p^2}{2}$ or $\frac{p^2}{2} + 1$ allows us to compute the majority on p^2 bits. There is an $\Omega(p^2) = \Omega(n^2)$ lower bound for majority, hence the same lower bound also holds for the computation of the maximal flow.

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