Introduction to big Ramsey degrees

Part 3: Ramsey degrees small and big, current work and open problems

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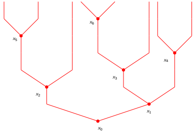
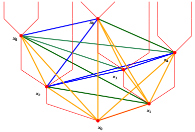
Recall: Laver–Devlin's proof of Big Ramsey degrees of rationals

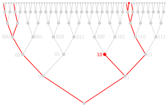
1. We well-ordered $\mathbb{Q}$ and produced tree of types



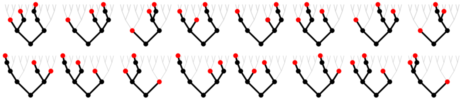
2. We a gave coloring of $\mathbb{Q}$ (by shapes of trees) so every copy of $\mathbb{Q}$ has "many colors"



3. We applied Milliken tree theorem to find copy of $\mathbb{Q}$ with "few colors"



4. We described colors as structures of compatible partial orders, so "few"="many"



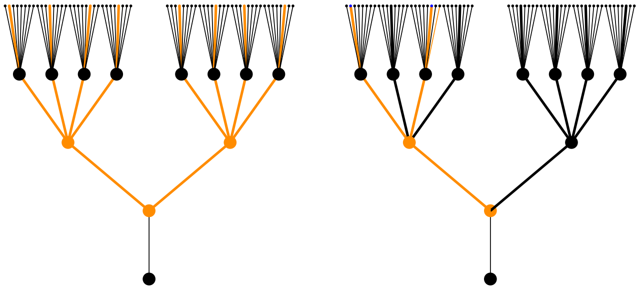
Recall: Trees with a successor operation

While most Ramsey-type theorems are concerned about regularly branching trees, we need more general notion allowing trees with finite but unbounded branching.

Definition ($\mathcal{S}$-tree)

An $\mathcal{S}$-tree is a quadruple $(T, \preceq, \Sigma, \mathcal{S})$ where $(T, \preceq)$ is a countable finitely branching tree with finitely many nodes of level 0, Σ is a set called the alphabet and $\mathcal{S}$ is a partial function $\mathcal{S}\colon T \times T^{<\omega} \times \Sigma \to T$ called the successor operation satisfying the following three axioms:

1. If $\mathcal{S}(a, \bar{p}, c)$ is defined for some base $a \in T$, parameter $\bar{p} \in T^{<\omega}$ and character $c \in \Sigma$, then $\mathcal{S}(a, \bar{p}, c)$ is an immediate successor of a and all nodes in $\bar{p}$ have levels at most $\ell(a) - 1$.
2. For every node $a \in T$ and its immediate successor b, there exist $\bar{p} \in T^{<\omega}$ and $c \in \Sigma$ such that $b = \mathcal{S}(a, \bar{p}, c)$.



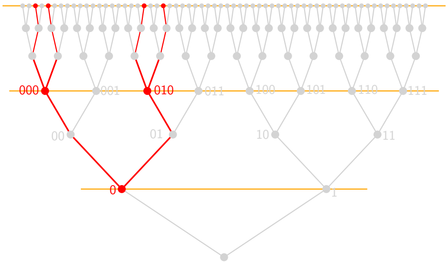
Recall: Shape-preserving functions

Definition (Shape-preserving functions)

Let $(T, \preceq, \Sigma, \mathcal{S})$ be an $\mathcal{S}$-tree. We call an injection $F\colon T \to T$ shape-preserving if

1. F is level preserving:
$$(\forall_{a,b\in T}) : (\ell(a) = \ell(b)) \implies (\ell(F(a)) = \ell(F(b)))$$
2. F is weakly $\mathcal{S}$-preserving:
$$(\forall_{a\in T, \bar{p}\in T^{<\omega}, c\in\Sigma}) : \mathcal{S}(a, \bar{p}, c) \text{ is defined } \implies \mathcal{S}(F(a), F(\bar{p}), c) \preceq F(\mathcal{S}(a, \bar{p}, c)).$$
3. For every $a \in T(0)$ and b such that $a \preceq b$ it also holds that $a \preceq F(b)$.

Given $S \subseteq T$, we also call a function $f\colon S \to T$ shape-preserving if it extends to a shape-preserving function $F\colon T \to T$.



Recall: Monoids of shape-preserving functions

For a level-preserving function $F\colon S \to T$, we denote by $\tilde{F}$ the function $\tilde{F}\colon \ell(S) \to \omega$ defined by $\tilde{F}(n) = \ell(F(a))$ for some $a \in S$ with $\ell(a) = n$.
We say that F is skipping level m if $m \notin \tilde{F}[\omega]$ and that F is skipping only level m if $\tilde{F}[\omega] = \omega \setminus \{m\}$.

Definition ($(\mathcal{S}, \mathcal{M})$-tree)

Given an $\mathcal{S}$-tree $(T, \preceq, \Sigma, \mathcal{S})$ and a monoid $\mathcal{M}$ of some shape-preserving functions $T \to T$, we call $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ an $(\mathcal{S}, \mathcal{M})$-tree if the following three conditions are satisfied:

1. $\mathcal{M}$ forms a closed monoid: $\mathcal{M}$ contains the identity and is closed for compositions and limits.
2. $\mathcal{M}$ admits decompositions: For every $n \in \omega$ and $F \in \mathcal{M}$ skipping level $\tilde{F}(n) - 1$ there exist $F_1, F_2 \in \mathcal{M}$ such that F_2 skips only level $\tilde{F}(n) - 1$ and $F_2 \circ F_1 \restriction_{T(\leq n)} = F \restriction_{T(\leq n)}$.
3. $\mathcal{M}$ is closed for duplication: For all n and m with $n < m \in \omega$, there exists a function $F^n_m \in \mathcal{M}$ skipping only level m such that for every $a \in T(n)$, $b \in T(m)$, $\bar{p} \in T^{<\omega}$ and $c \in \Sigma$, where $\mathcal{S}(a, \bar{p}, c)$ is defined and $\mathcal{S}(a, \bar{p}, c) \preceq b$, we have $F^n_m(b) = \mathcal{S}(b, \bar{p}, c)$.

Recall: Topological Ramsey theorem for trees with successor operation

Theorem (Balko, Dobrinen, Chodounský, H., Konečný, Nešetřil, Zucker day before yesterday)

Let $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ be an $(\mathcal{S}, \mathcal{M})$-tree and consider $\mathcal{M}$ with the Ellentuck topology. Then every property of Baire subset of $\mathcal{M}$ is Ramsey and every meager subset is Ramsey null.

Proof outline

1. 1-dimensional pigeonhole is proved using Hales–Jewett theorem (duplication is important here).

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We obtain an interesting example of Ramsey space where Todorčević A3.2 axiom is not satisfied.

Application to free amalgamation classes

Theorem (Zucker 2022)

Let L be a finite binary language and $\mathcal{F}$ a finite family of irreducible L-structures. Then every countable universal $\mathcal{F}$-free structure has finite big Ramsey degrees.

L denotes an language containing of relational symbols. We consider standard model-theoretic L-structures.

An L-structure $\mathbf{A}$ is irreducible if for every pair of vertices u, v there exists $R \in L$ and $\bar{x} \in R_{\mathbf{A}}$ containing both u and v. (This is a generalization of a graph clique.)

An L-structure $\mathbf{A}$ is $\mathcal{F}$-free if there is no $\mathbf{F} \in \mathcal{F}$ that embeds to $\mathbf{A}$.

$\mathcal{F}$-free L-structure $\mathbf{A}$ is universal if every countable $\mathcal{F}$-free L-structure embeds to $\mathbf{A}$.

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Proofs of upper bounds on Big Ramsey degrees with non-trivial forbidden substructures:

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5. M. Balko, D. Chodounský, N. Dobrinen, J. H., M. Konečný, J. Nešetřil, A. Zucker: Ramsey theorem for trees with successor operation. arXiv:2311.06872 (2023+). 9 out of 37 pages. Using general tree theorem proved by combinatorial forcing and Todorčević' Ramsey Space axioms. All enumeration trees.

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Reverse mathematics:

1. P.-E. Anglés d'Auriac, P. A. Cholak, D. D. Dzhafarov, B. Monin, L. Patey: Milliken's tree theorem and its applications: a computability-theoretic perspective. To appear in Memoirs of AMS (2020–). 174 pages.
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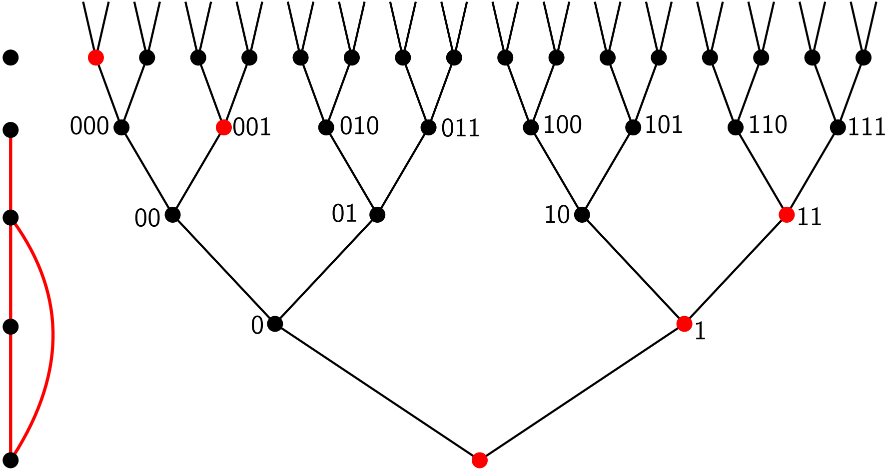
Coding tree (Dobrinen, Zucker)



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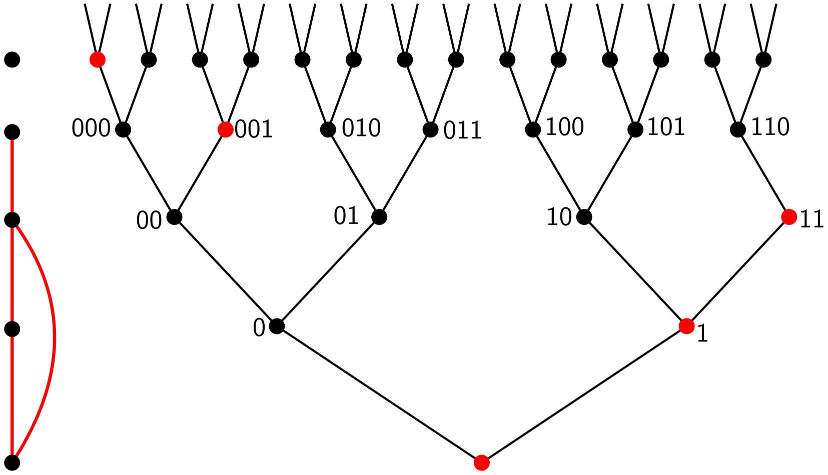
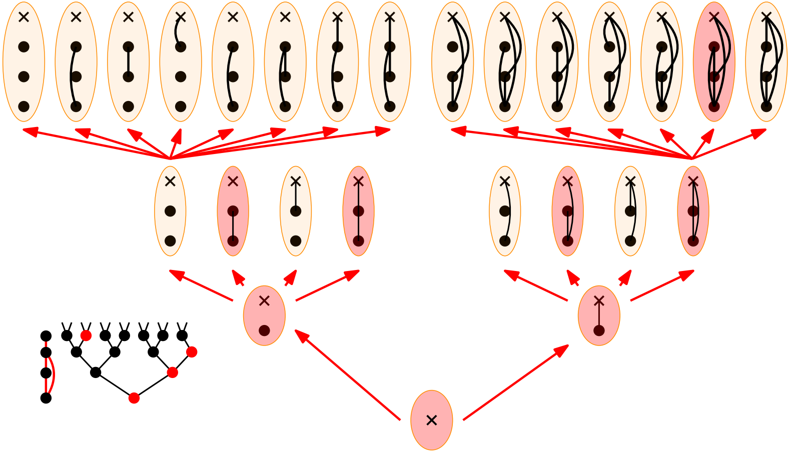


All enumerations tree



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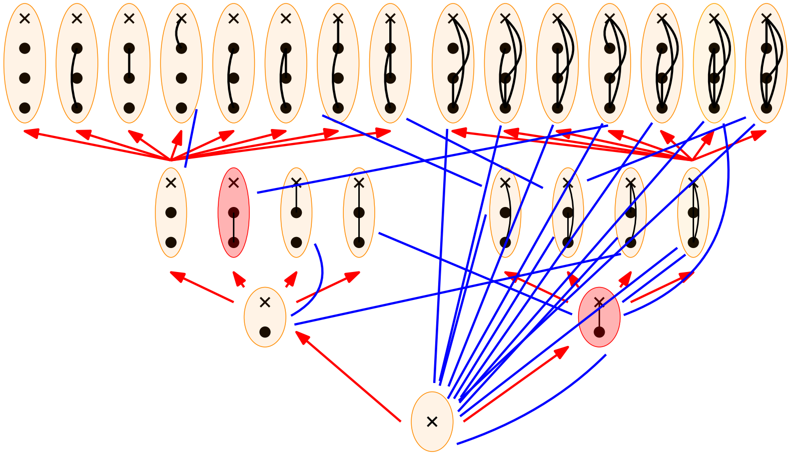

Constructing all enumeration tree

Definition (Type)

Type of level n is an $\mathcal{F}$-free L-structure $\mathbf{A}$ with vertices $\{0, 1, \ldots, n-1, t\}$, where t is the type vertex.

Definition (Levelled type)

Levelled type of level n is a pair $\mathbf{a} = (\mathbf{A}, \mathrm{fl}_{\mathbf{A}})$ where $\mathbf{A}$ is a type of level n and $\mathrm{fl} : n \setminus \{0\} \to n$ is a function satisfying:

1. $\mathrm{fl}_{\mathbf{a}}(i) < i$.
2. whenever $i < j$ forms an edge of $\mathbf{A}$ then $\mathrm{fl}_{\mathbf{A}}(j) > i$.

Nodes of an $\mathcal{S}$-tree are levelled types ordered by inclusion. Successor operation is an amalgamation.

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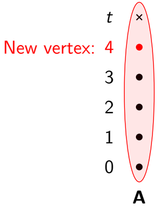

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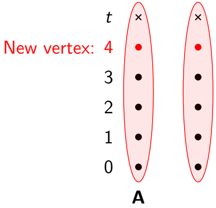

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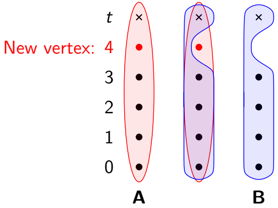

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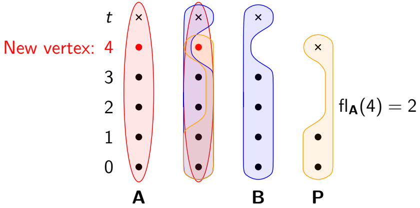

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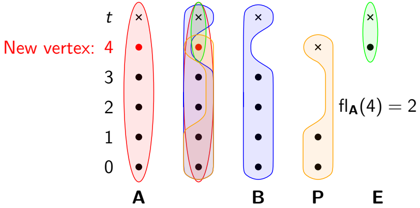

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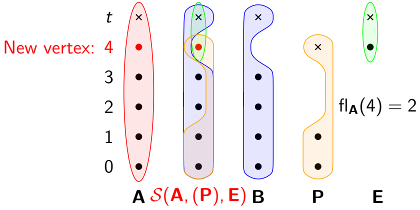

Non-forcing proof of Zucker's theorem (sketch)

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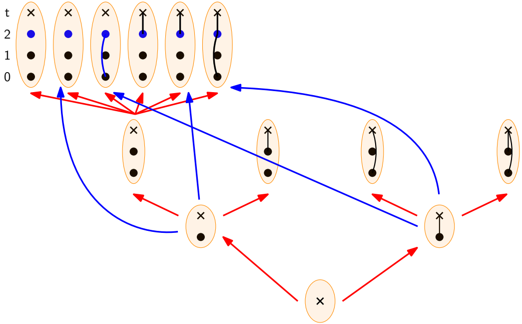

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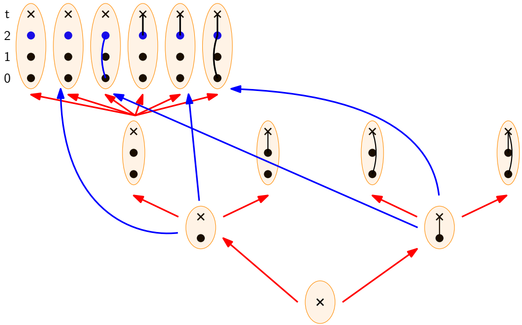

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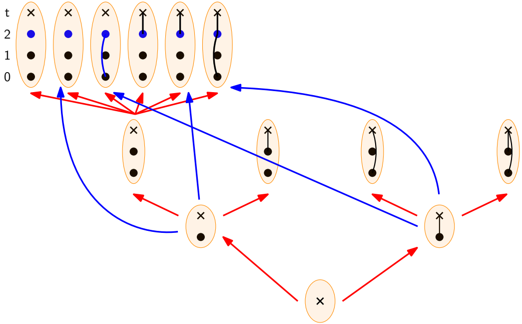

2. Verify decomposition and duplication on the monoid of all shape-preserving functions.
3. Define structure on nodes of the $\mathcal{S}$-tree and verify that shape-preserving functions preserve the structure
4. Verify that envelopes are bounded for nice copies inside nice enumerations (same was as in Zucker's paper)

More general result

Theorem

Let L be a finite language consisting of unary and binary symbols, and let $\mathcal{K}$ be a hereditary class of finite structures and $k \geq 2$. Assume that every countable structure **A** *has a completion to $\mathcal{K}$ provided that every induced cycle in* **A** *(seen as a substructure) has a completion in $\mathcal{K}$ and every irreducible substructure of* **A** *of k embeds into $\mathcal{K}$. Then $\mathcal{K}$ has a Fraïssé limit with finite big Ramsey degrees.*

This result can be used to analyze all Cherlin's catalogues of binary homogeneous structures except for those described by infinitely many forbidden cliques (Henson graphs).

Connection to finite structural Ramsey theory

Ramsey classes

Definition

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{2}, 1$.

$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of structure **A** to structure **B**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

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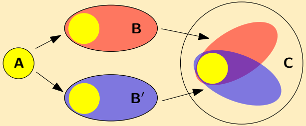
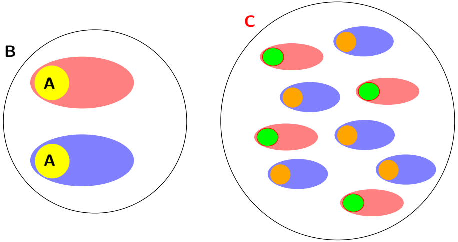
Ramsey classes are amalgamation classes

Definition (Amalgamation)



Nešetřil, 80's: Under mild assumptions Ramsey classes have amalgamation property.



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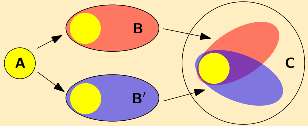

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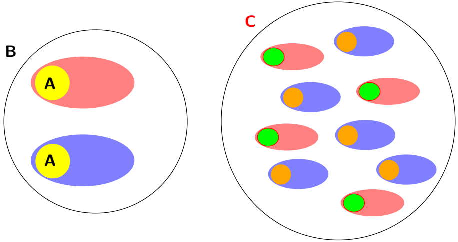

Zucker 2020: All known big Ramsey degrees can be turned into a big Ramsey structure.
Ages of big Ramsey structures have amalgamation property.

Fraïssé theory

Definition (Age)

Given structure **A** its age, Age(**A**), is the set of all finite structures with embedding to **A**.

Definition (Homogeneity)

Structure **H** is homogeneous if every isomorphism of its two finite (induced) substructures (a partial automorphism of **H**) extends to an automorphism of **H**.

Theorem (Fraïssé, 1950s)

A hereditary, isomorphism-closed class $\mathcal{K}$ with countably many mutually non-isomorphic structures is an age of a homogeneous structure **A** *if and only if it has the amalgamation property.*

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By Nešetřil's observation ages of homogeneous structure are candidates for Ramsey classes

The Lachlan–Cherlin classification programme of homogeneous structures

Theorem (Schmerl 1979)

Every homogeneous partial order is isomorphic to one of the following:

1. *a (possibly infinite) antichain,*
2. *a (possibly infinite) antichain of chains,*
3. *chain of antichains,*
4. *the generic partial order.*

Theorem (Lachlan 1984)

Homogeneous tournaments are:

1. *finite cases: one-point tournament, oriented cycle of length 3.*
2. $(\mathbb{Q}, \leq)$
3. *dense local order,*
4. *generic tournament.*

Theorem (Lachlan–Woodrow 1980)

Let G be an (countably) infinite homogeneous graph. Then either G or its complement is isomorphic to one of:

1. *Rado graph (universal homogeneous graph),*
2. *universal homogeneous graph omitting complete graphs of size n,*
3. *a disjoint union of complete graphs, all of same size.*

Classification of countable homogeneous digraphs (Cherlin, 1998)

...

Metricaly homogeneous graphs (Cherlin, 2023+)

KPT-correspondence and Nešetřil's classification Programme

Nešetřil's classification Programme

$$\begin{array}{ccc} \text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\ \Uparrow & & \Downarrow \\ \text{expansions of homogeneous} & \Longleftarrow & \text{homogeneous structures} \end{array}$$

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Ramsey classes	$\Longrightarrow$	amalgamation classes
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expansions of homogeneous	$\Longleftarrow$	homogeneous structures
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extremely amenable groups	$\Longrightarrow$	universal minimal flows

Theorem (Kechris–Pestov–Todorčević 2005: KPT-correspondence)

The group of automorphisms of the Fraïssé limit of a amalgamation class $\mathcal{K}$ is extremely amenable if and only if $\mathcal{K}$ is a Ramsey class.

KPT-correspondence and Nešetřil's classification Programme

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Let L' be language containing language L. A expansion of L-structure $\mathbf{A}$ is L'-structure $\mathbf{A}'$ on the same vertex set such that all relations/functions in $L \cap L'$ are identical.

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Theorem (Nešetřil, 1989)

"All homogeneous graphs have Ramsey expansion."

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Theorem (Jasiński–Laflamme–Nguyen Van Thé–Woodrow, 2014)

All homogeneous digraphs have precompact Ramsey expansions with expansion property.

Today all catalogues of homogeneous structures have corresponding Ramsey classes.

Classification Programme of big Ramsey structures

1. Structures with finite big Ramsey degrees may lead to big Ramsey structures. This is (in all known cases) the minimal expansion of original structure with big Ramsey degree 1.

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It turns out that homogeneous structures are not optimal setup for classifying big Ramsey degrees. The main difference is that all big Ramsey structures arise from well-ordering and thus they are not homogeneous. Better formalism is work in progress.

A. Aranda, S. Braunfeld, D. Chodounský, H., M. Konečný, J. Nešetřil, and A. Zucker: Type-respecting amalgamation and big Ramsey degrees, extended abstract in EUROCOMB 2023

Big Ramsey degrees contributing to finite structural Ramsey theory

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4. Proofs of upper bounds on big Ramsey degrees are more laborious, but also more systematic. Tree of types is not visible to small Ramsey problems, but in a way still present.

Simplified formulation for regularly branching trees

Definition (Boring extensions)

Given finite alphabet Σ, a family of boring extensions is a countable sequence $\mathcal{E} = (\mathcal{E}_n)_{n\in\omega}$ of sets

$$\mathcal{E}_n \subseteq \{e \text{ is a function } e\colon \Sigma^n \to \Sigma\}$$

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2. Insertion: For every $m \leq n$, $e_1 \in \mathcal{E}_m$, $e_2 \in \mathcal{E}_n$ there exists $e_3 \in \mathcal{E}_{n+1}$ such that for every $a \in \Sigma^m$ and $b \in \Sigma^{n-m}$ the following is satisfied:

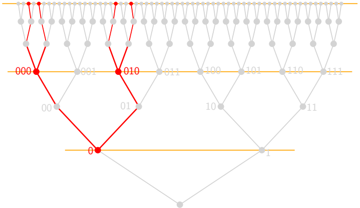
$$e_3(a^\frown e_1(a)^\frown b) = e_2(a^\frown b).$$

Generalized embedding types

Definition (Interesting levels)

Given a finite alphabet Σ, a family of ~~boring~~ extensions $\mathcal{E}$ and a set $X \subseteq \Sigma^{<\omega}$ we denote by $I_{\mathcal{E}}(X)$ the set of interesting levels of X. This is a set of all $\ell \in \omega$ such that there is no $e_\ell \in \mathcal{E}_\ell$ satisfying for every $a \in X$, $|a| \geq \ell$

$$(a|_\ell)^\frown e_\ell(a|_\ell) \sqsubseteq a.$$



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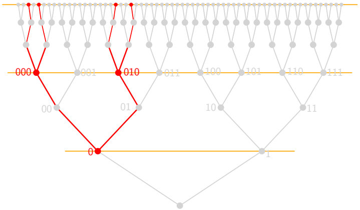

Definition (Embedding type)

Given a finite alphabet Σ, a family of ~~boring~~ extensions $\mathcal{E}$ and a set $X \subseteq \Sigma^{<\omega}$ we define the embedding type of X, denoted $\tau_{\mathcal{E}}(X)$, to be the set of all words created from words in X by removing all letters with indices not in $I_{\mathcal{E}}(X)$.

Coloring subsets of a given embedding type

Given a finite alphabet Σ, a family of ~~boring~~ extensions $\mathcal{E}$ and sets $X, Y \subseteq \Sigma^{<\omega}$ we put

$$\binom{Y}{X/\mathcal{E}} = \{X' \subseteq Y : \tau_{\mathcal{E}}(X') = \tau_{\mathcal{E}}(X)\}.$$

Theorem (BCDHKNZ 2023+, Colouring sets of given embedding type in a finite tree)

For every finite alphabet Σ, family of ~~boring~~ extensions $\mathcal{E}$, positive integer r, finite set $X \subseteq \Sigma^{<\omega}$ and (possibly infinite) set $Y \subseteq \Sigma^{<\omega}$ and every finite colouring χ of $\binom{\Sigma^{<\omega}}{X/\mathcal{E}} \to r$ there exists $Y' \in \binom{\Sigma^{<\omega}}{Y/\mathcal{E}}$ such that χ is constant when restricted to $\binom{Y'}{X/\mathcal{E}}$.

Theorem (BCDHKNZ 2023+, Colouring sets of given embedding type in a combinatorial cube)

For every finite alphabet Σ, family of ~~boring~~ extensions $\mathcal{E}$, positive integers m, n, r and (finite) sets $X \subseteq \Sigma^m$, $Y \subseteq \Sigma^n$ there exists $N \in \omega$ such that for every r-colouring $\chi\colon \binom{\Sigma^N}{X/\mathcal{E}} \to r$ there exists $Y' \in \binom{\Sigma^N}{Y/\mathcal{E}}$ such that χ is constant when restricted to $\binom{Y'}{X/\mathcal{E}}$.

Abramson–Harrington theorem

Let L be a relational language. We consider standard model-theoretic L-structures.

If L contains binary symbol $\leq$ we call L-structure $\mathbf{A}$ ordered if $(A, \leq_{\mathbf{A}})$ is a linear order.

Given L-structures $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ we write $\binom{\mathbf{B}}{\mathbf{A}}$ for the set of all embeddings $\mathbf{A} \to \mathbf{B}$.

Definition (Partition arrow)

We write $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ for the following statement:

> *For every 2-coloring $\chi : \binom{\mathbf{C}}{\mathbf{A}} \to \{0, 1\}$ there exists embedding $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that χ restricted to $\{f \circ g : g \in \binom{\mathbf{B}}{\mathbf{A}}\}$ is constant.*

Theorem (Nešetřil 1977, Abramson–Harrington 1978)

Let L be a relational language and $\mathbf{A}, \mathbf{B}$ finite ordered L-structures. Then there exists finite ordered L-structure $\mathbf{C}$ satisfying $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

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Proof, step 1: associate vertices of structure B with words.

1. Fix $\mathbf{A}$ and $\mathbf{B}$. WLOG assume that $B = n = |B|$ and $\leq_{\mathbf{B}}$ is the natural ordering of n.

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3. Put $p = 2^n - 1$ and enumerate all non-empty substructures of $\mathbf{B}$ as $\mathbf{B}^0, \mathbf{B}^1, \ldots, \mathbf{B}^{p-1}$ in the increasing order (given by $\preceq$). For each $i < p$

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4. For each $i < N$ find lexicographically first substructure $\mathbf{D}^i$ isomorphic to $\mathbf{B}^i$ and denote by f^i the unique isomorphism $\mathbf{B}^i \to \mathbf{D}^i$.

$$\varphi(v)_i = \begin{cases} -1 & \text{if } v \notin B^i \\ f^i(v) & \text{if } v \in B^i \end{cases} \quad \text{for every } v \in B \text{ and } i < p.$$

B 

		0	1	2	3	4	5	6
0	$\varphi(0) =$	0	n	n	0	0	n	0
1	$\varphi(1) =$	n	0	n	1	n	0	1
2	$\varphi(2) =$	n	n	0	n	2	1	2

□

Abramson–Harrington theorem

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Proof, step 2: structure $\mathbf{C}_\ell$ on Σ^ℓ.

Consider regularly branching tree $(\Sigma^{<\omega}, \sqsubseteq)$ with $\Sigma = B \cup \{-1\}$.

Abramson–Harrington theorem

			0	1	2	3	4	5	6
	0	$\varphi(0) =$	0	n	n	0	0	n	0
B	1	$\varphi(1) =$	n	0	n	1	n	0	1
	2	$\varphi(2) =$	n	n	0	n	2	1	2

Proof, step 2: structure $\mathbf{C}_\ell$ on Σ^ℓ.

Consider regularly branching tree $(\Sigma^{<\omega}, \sqsubseteq)$ with $\Sigma = B \cup \{-1\}$.
Given $k, \ell \in \omega$, and a tuple $\bar{w} = (w^0, w^1, \ldots, w^{k-1})$ of elements of Σ^ℓ

1. we say that $\bar{w}$ decides a structure on level $i < \ell$ if $0 \leq w_i^0 < w_i^1 < \cdots < w_i^{k-1}$ and i is a minimal with this property.

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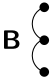

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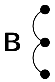
2. we say that $\bar{w}$ become incompatible on level $i' < \ell$ if either
   1. $k = 2$ and $w^0_{i'} \geq w^1_{i'} \geq 0$,
   2. $0 \leq w^0_{i'} < w^1_{i'} < \cdots < w^{k-1}_{i'}$ however there exists $i < i'$ such that $\bar{w}$ decides structure on level i and $B \restriction_{\{w^0_i, w^1_i, \dots, w^{k-1}_i\}}$ is not isomorphic to $B \restriction_{\{w^0_{i'}, w^1_{i'}, \dots, w^{k-1}_{i'}\}}$.

Abramson–Harrington theorem



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For every $\ell \in \omega$ construct an ordered L-structure $\mathbf{C}_\ell$ as a structure satisfying the following:

1. The vertex set of $\mathbf{C}_\ell$ is $C_\ell = \Sigma^\ell$,
2. $\leq_{\mathbf{C}_\ell}$ is the lexicographic ordering of Σ^ℓ,
3. whenever $(w^0, w^1, \ldots, w^{k-1}) \in \Sigma^\ell$ is compatible and decides structure on some level i then $B \restriction_{\{w^0, w^1, \ldots, w^{k-1}\}}$ is isomorphic to $B \restriction_{\{w^0_i, w^1_i, \ldots, w^{k-1}_i\}}$.

Abramson–Harrington theorem

Proof step 3: Building $(\mathcal{S}, \mathcal{M})$-tree.

Define successors by concatenation.
Let $\mathcal{M}$ denote the set of all shape-preserving functions $F\colon \Sigma^{<\omega} \to \Sigma^{<\omega}$ satisfying for every $\ell \in \omega$ and every lexicographically increasing sequence $\bar{w}$ of elements of Σ^{ℓ} the following two properties:

1. if $F(\bar{w})$ decides structure on level i then $i \in \tilde{F}[\omega]$.
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Let N by given by our theorem for $(\mathcal{S}, \mathcal{M})$-tree, $2^{|A|} - 1$ and $2^{|B|} - 1$. Then

$$\mathbf{C}_\ell \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

□

B

		0	1	2	3	4	5	6
0	$\varphi(0) =$	0	n	n	0	0	n	0
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A

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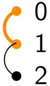
B 

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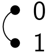
A 

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B (0, 1, 2)

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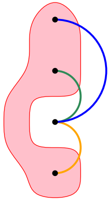
$\varphi(1) =$	~~n~~	0	n	~~1~~	~~n~~	0	~~1~~
$\varphi(2) =$	~~n~~	n	0	~~n~~	~~2~~	1	~~2~~

A (0, 1)

	0	1	2
$\varphi(0) =$	0	n	0
$\varphi(1) =$	n	0	1

Open problems

1. Does the class of all finite structures omitting the following substructure have finite big Ramsey degrees?



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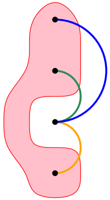

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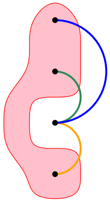

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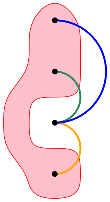

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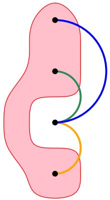

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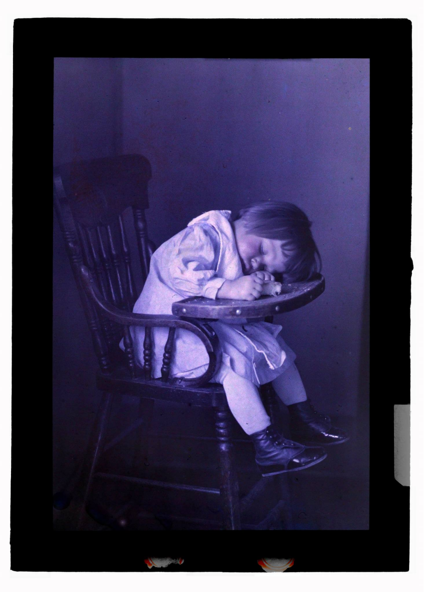
See list of open problems in:
M. Konečný: Model theory and extremal combinatorics, PhD thesis, 2023.
`https://kam.mff.cuni.cz/~matej/`

Thank you for the attention



Sleeping Child, Fred Payne Clatworthy, Autochrome, 7 x 5 inches, c1916
Mark Jacobs Collection