Introduction to big Ramsey degrees

Part 2: graphs and restricted graphs

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Recall

Theorem ((Infinite) Ramsey Theorem, 1930)

$$\forall_{p,k \geq 1} : \omega \longrightarrow (\omega)^{p}_{k,1}.$$

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Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}} \exists_{T=T(|O|)\in\omega} \forall_{k\geq 1} : (\mathbb{Q},\leq) \longrightarrow (\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

$T(n)$ is the big Ramsey degree of n tuple in $\mathbb{Q}$.

$$T(n) = \tan^{(2n-1)}(0).$$

$$T(1) = 1, T(2) = 2, T(3) = 16, T(4) = 272,$$
$$T(5) = 7936, T(6) = 353792, T(7) = 22368256$$

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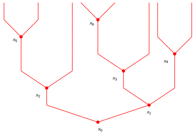
The proof (due to Laver) makes essential use of the Milliken tree theorem. This proof may seem bit arbitrary. However trees are essential (arise naturally as rich colourings). Precise bounds can be understood as a justification that this is the only approach.

Story so far

1. We well-ordered $\mathbb{Q}$ and produced tree of types



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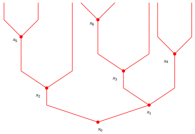
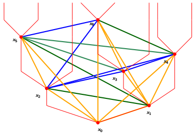


2. We a gave coloring of $\mathbb{Q}$ (by shapes of trees) so every copy of $\mathbb{Q}$ has “many colors”



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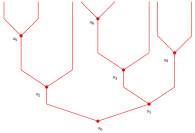

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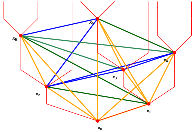

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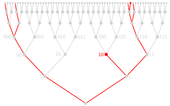

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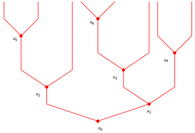

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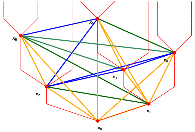

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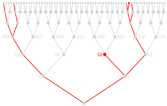
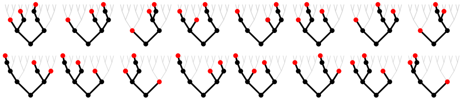
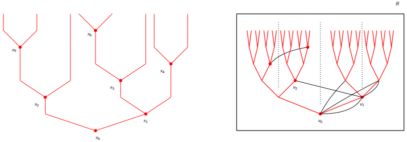


4. We described colors as structures of compatible partial orders, so “few”=“many”

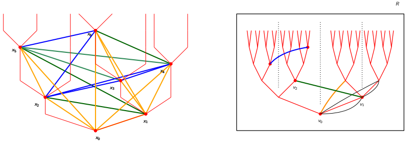


Big Ramsey degrees of the Rado graph

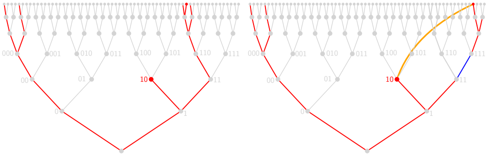
1. Enumerate **R** and produce tree of types



2. Give a coloring of **R** (by shapes of trees) so every copy of **R** has “many colors”



3. Apply Milliken tree theorem to find copy of **R** with “few colors”



4. Describe minimal set of colors as structures, so “few”=“many”

Some more recent results on big Ramsey degrees

1. Nguyen Van Thé (2009): Characterisation of big Ramsey degrees of homogeneous ultrametric spaces
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3. Dobrinen (2020): Big Ramsey degrees of universal homogeneous triangle-free graphs are finite
4. Dobrinen (2023): Big Ramsey degrees of universal homogeneous K_k-free graphs are finite for every $k \geq 3$.
5. Zucker (2022): Big Ramsey degrees of Fraïssé limits of free amalgamation classes in binary language with finitely many forbidden substructures are finite.
6. Balko, Chodounský, H., Konečný, Vena (2022): Big Ramsey degrees of 3-uniform hypergraphs are finite.
7. H. (2020+): Big Ramsey degrees of partial orders and metric spaces are finite.
8. Balko, Chodounský, Dobrinen, H., Konečný, Nešetřil, Vena, Zucker (2021): Big Ramsey degrees of structures described by induced cycles are finite.
9. Balko, Chodounský, Dobrinen, H., Konečný, Vena, Zucker (2021+): Characterisation of big Ramsey degrees of Fraïssé limits of free amalgamation classes in binary language with finitely many constraints.
10. Bice, de Rancourt, H., Konečný: metric big Ramsey degrees of ℓ_∞ and the Urysohn sphere, (2023+).

Big Ramsey degrees of restricted structures

Let $\mathcal{G}_3$ be the class of all finite triangle-free graphs.

Theorem (Dobrinen 2020)

Every (countable) universal triangle-free graph $\mathbf{R}_3$ *has finite big Ramsey degrees:*

$$\forall_{\mathcal{A}\in\mathcal{G}_3}\exists_{T=T(|\mathbf{A}|)\in\omega}\forall_{k\geq 1} : \mathbf{R}_3 \longrightarrow (\mathbf{R}_3)^{(\mathbf{A})}_{k,T}.$$

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Let $\mathcal{P}$ be the class of all finite partial orders.

Theorem (J. H. 2020+)

Every (countable) universal partial order $(P,\leq)$ *has finite big Ramsey degrees:*

$$\forall_{(O,\leq)\in\mathcal{P}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (P,\leq) \longrightarrow (P,\leq)^{(O,\leq)}_{k,T}.$$

Universality: every countable partial order has embedding to $(P,\leq)$.

Tree of types of a universal triangle-free graph

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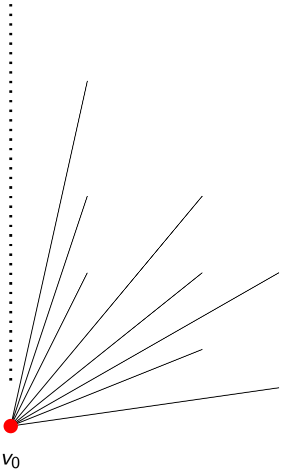

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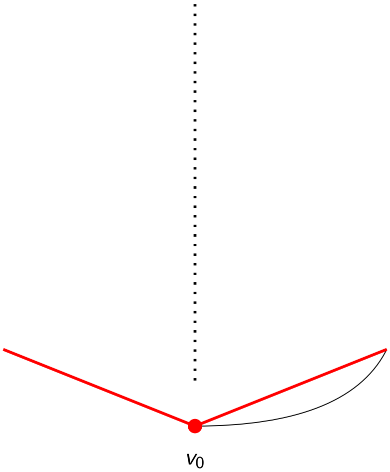

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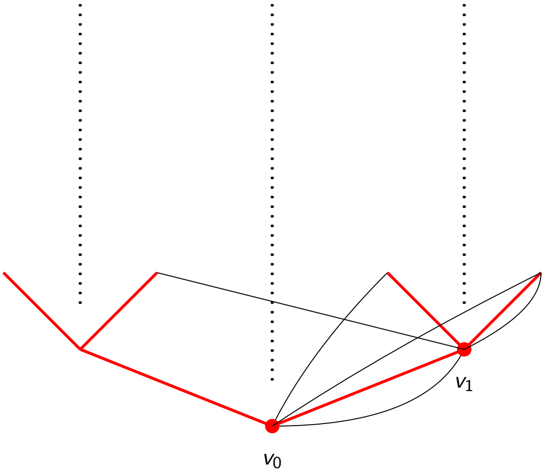

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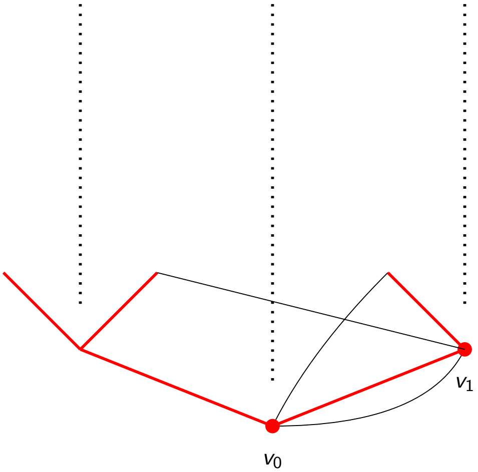

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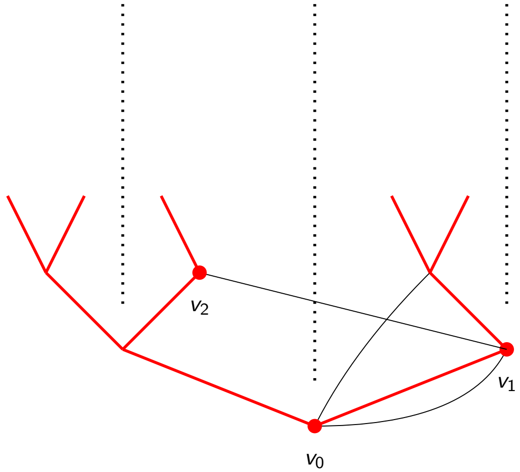

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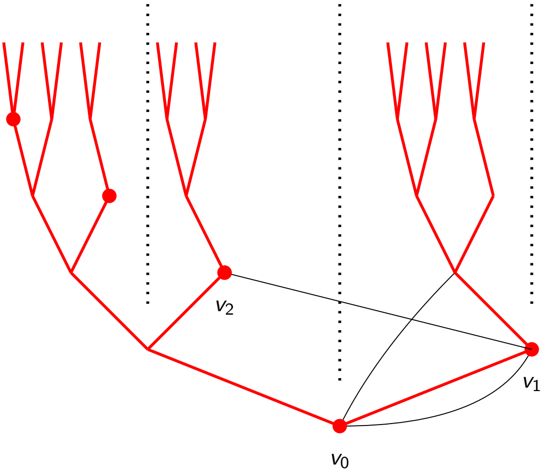

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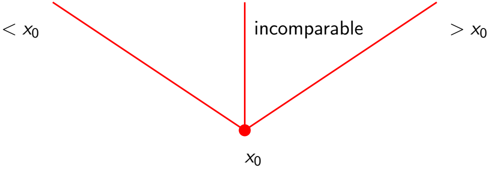

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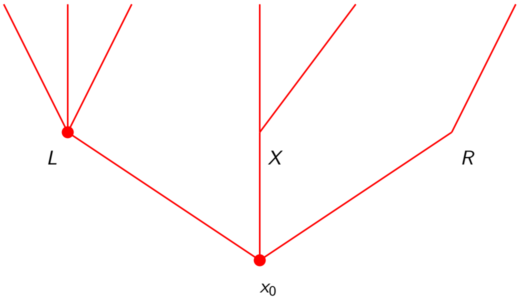

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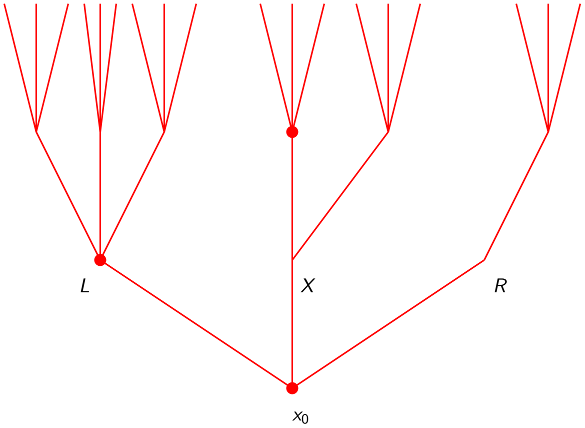

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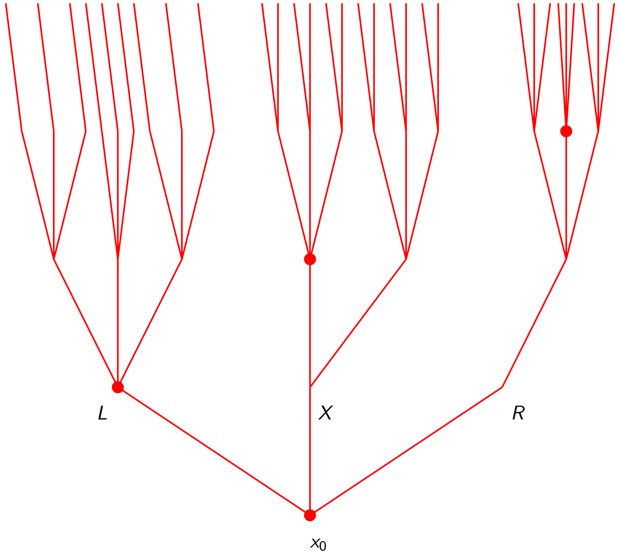

Parameter words

Definition (Parameter word)

Given a finite alphabet Σ and $k \in \omega + 1$, a *k-parameter word* is a (possibly infinite) word W in alphabet $\Sigma \cup \{\lambda_i \colon 0 \leq i < k\}$ such that $\forall i \in k$ word W contains λ_i and for every $j \in k - 1$, the first occurrence of λ_{j+1} appears after the first occurrence of λ_j.

Example (2-parameter word)

$\Sigma = \{\mathsf{L}, \mathsf{X}, \mathsf{R}\}$.

$$\mathsf{LRL}\lambda_0\lambda_0\mathsf{X}\lambda_1\lambda_0\mathsf{R}$$

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For set S of parameter words and a parameter word W:

$$W(S) = \{W(U) : U \in S\}.$$

Ramsey theorem for parameter words

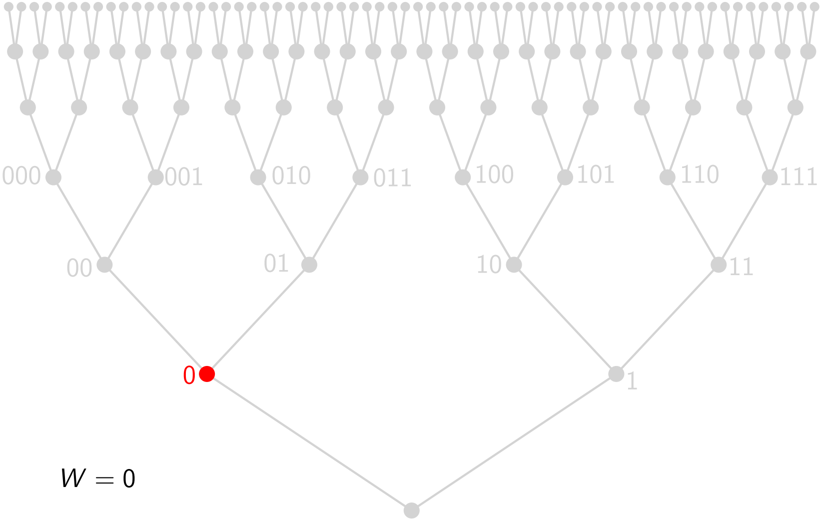
The following infinitary version of Graham–Rothschild Theorem is a direct consequence of the Carlson–Simpson theorem. It was also independently proved by Voight in 1983 (apparently unpublished):

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

By W being monochromatic we mean that for every pair of k-parameter words U, V the colour of $W(U)$ is the same as colour of $W(V)$.

Parameter words as subtrees



$W = 0$

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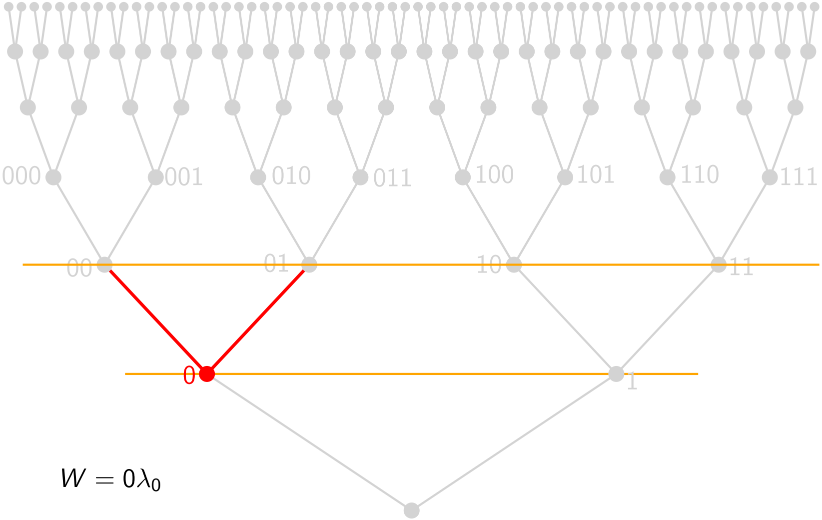

$W = 0\lambda_0$

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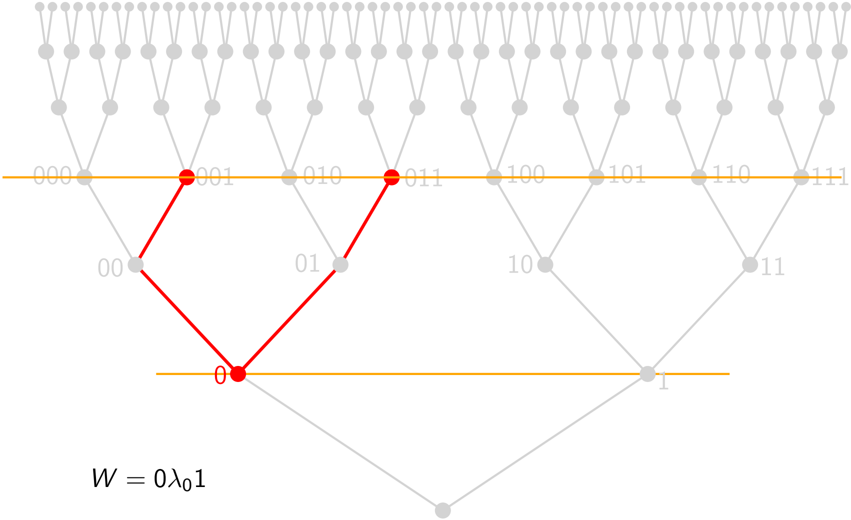

$W = 0\lambda_0 1$

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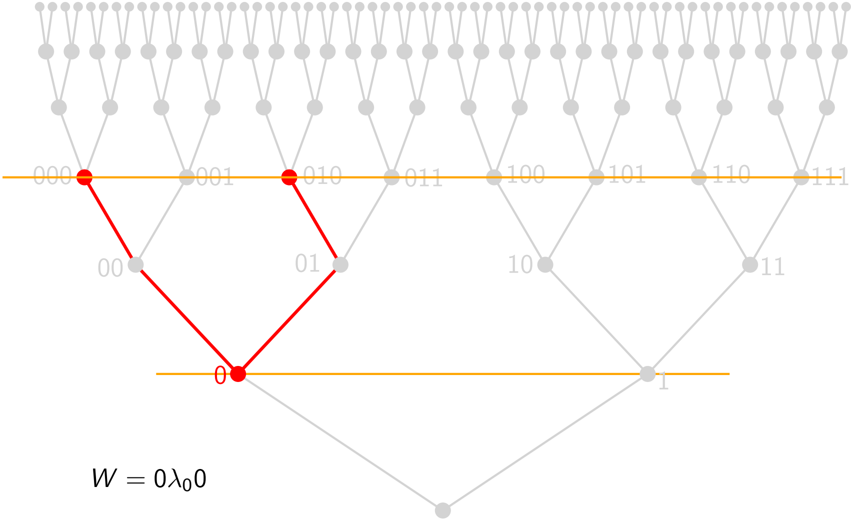

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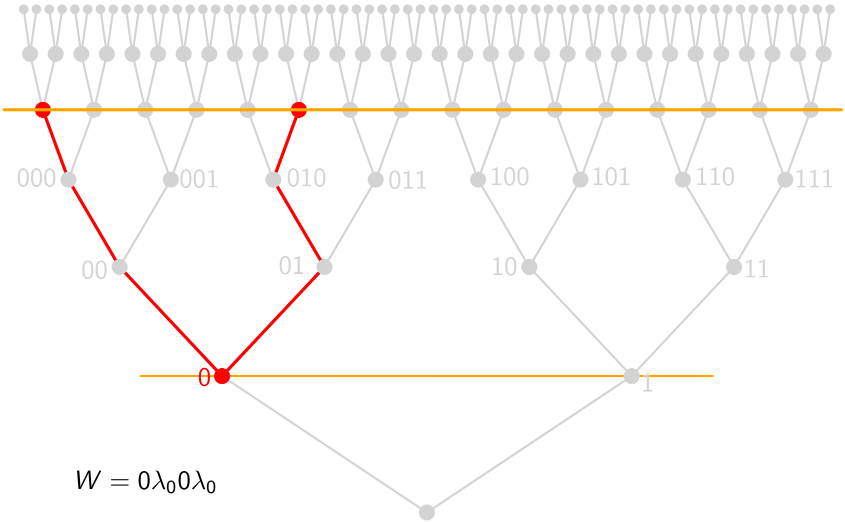

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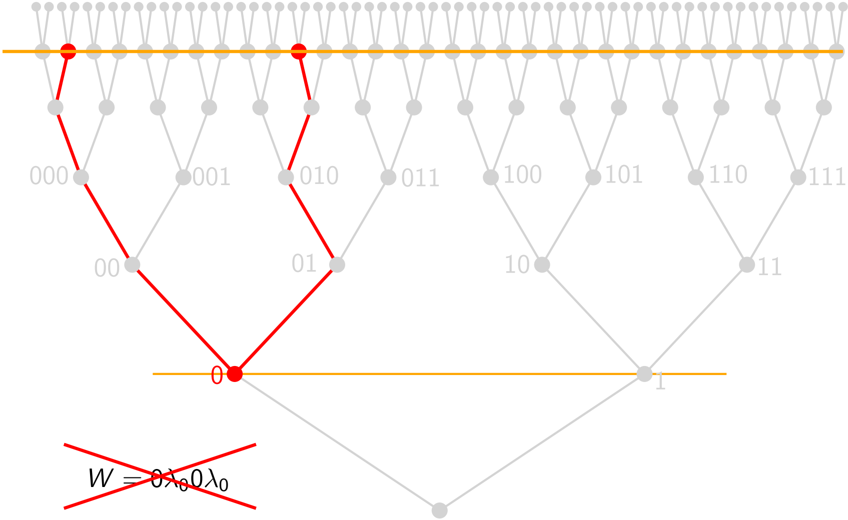

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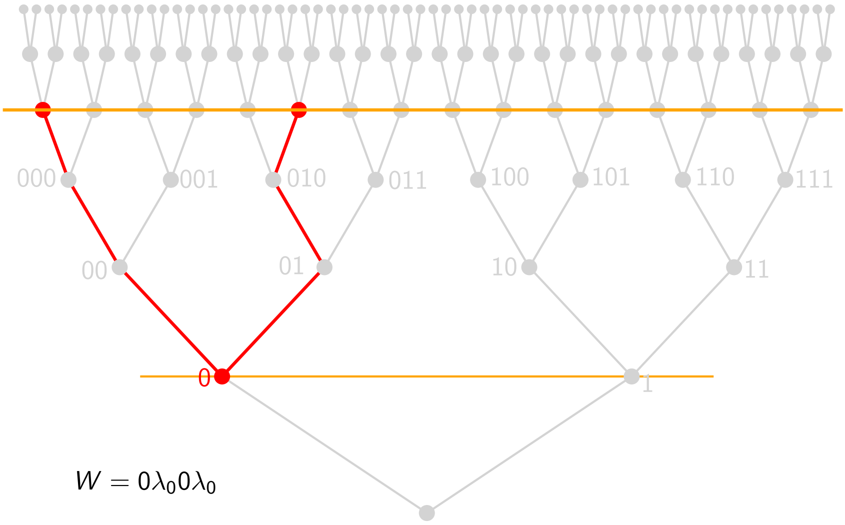

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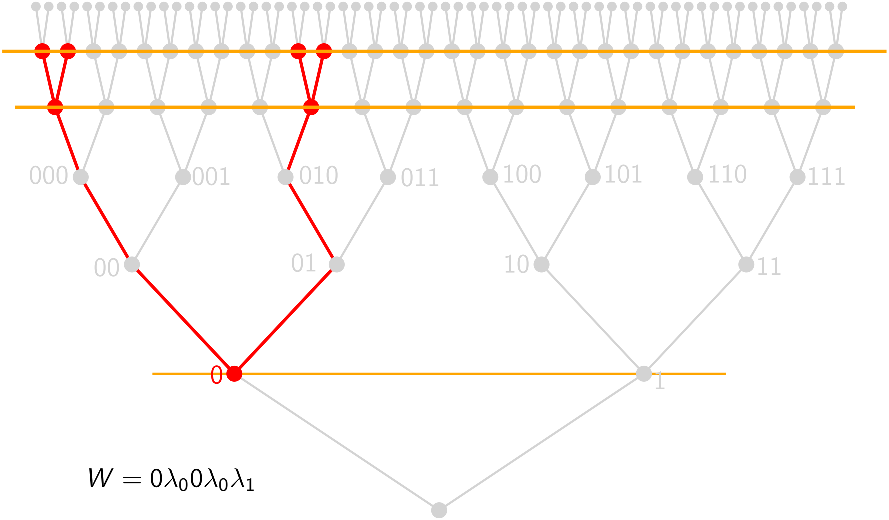

$W = 0\lambda_0 0\lambda_0 \lambda_1$

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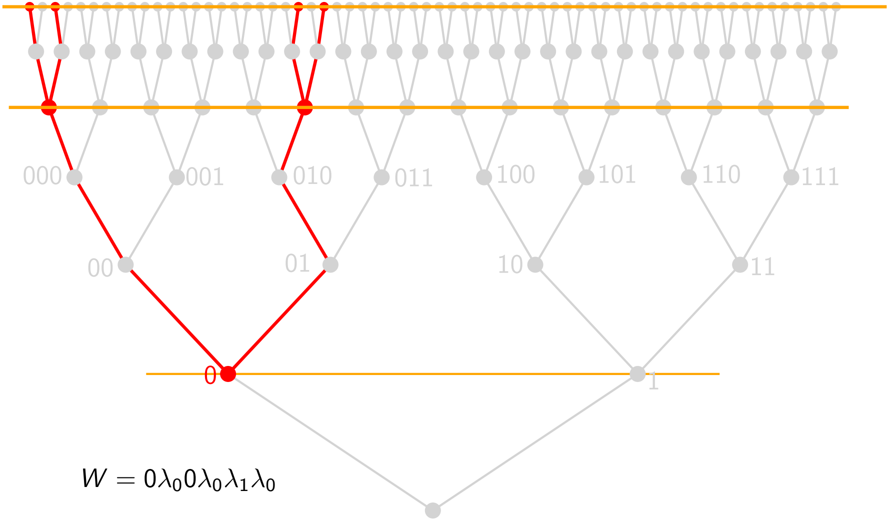

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Definition

Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set k-parameter words, an envelope of S is every n-parameter word W (for some $n \geq k$) such that

$$\forall_{w \in S} \exists_u W(u) = w.$$

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Example

Envelopes of $\{0, 000\}$ are: $0\lambda_0\lambda_0$, $0\lambda_0 0$, $0\lambda_0 0\lambda_1$, $\dots$.

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Proposition (Envelopes are bounded)

Let Σ be a finite alphabet, and $k, s \geq 0$ be a finite integers. Then there exists $T(|\Sigma|, s, k)$ such that for every set S of size s of k-parameter words in alphabet Σ there exists an envelope of S with at most $T(|\Sigma|, s, k)$ parameters.

Proof.

$$\begin{array}{llllllll} U & = & 0 & 1 & 1 & 0 & 1 & \\ V & = & 0 & 0 & 1 & 1 & 0 & 1 \end{array}$$

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Envelopes of $\{0, 000\}$ are: $0\lambda_0\lambda_0$, $0\lambda_0 0$, $0\lambda_0 0\lambda_1$, $\ldots$.

Proposition (Envelopes are bounded)

Let Σ be a finite alphabet, and $k, s \geq 0$ be a finite integers. Then there exists $T(|\Sigma|, s, k)$ such that for every set S of size s of k-parameter words in alphabet Σ there exists an envelope of S with at most $T(|\Sigma|, s, k)$ parameters.

Proof.

U	$=$	0	1	1	0	1	
V	$=$	0	0	1	1	0	1
Envelope:		0	λ_0	1	λ_1	λ_0	

Definition

Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set k-parameter words, an envelope of S is every n-parameter word W (for some $n \geq k$) such that

$$\forall_{w \in S} \exists_u W(u) = w.$$

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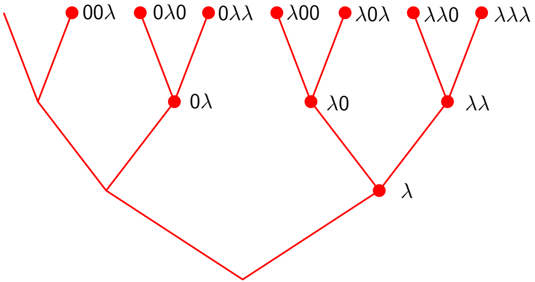
Proof.

$$\begin{array}{ccccccc}
U & = & 0 & 1 & 1 & 0 & 1 \\
V & = & 0 & 0 & 1 & 1 & 0 & 1 \\
\text{Envelope:} & & 0 & \lambda_0 & 1 & \lambda_1 & \lambda_0 & \lambda_2
\end{array}$$

□

Triangle-free graph on 1-parameter words



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Definition (Triangle-free graph G)

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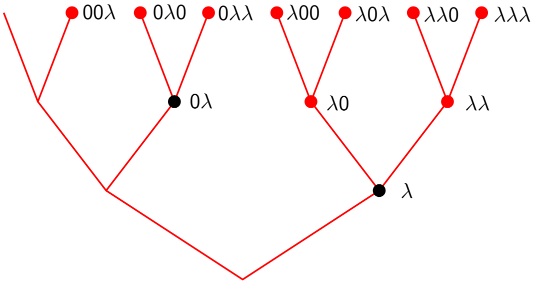

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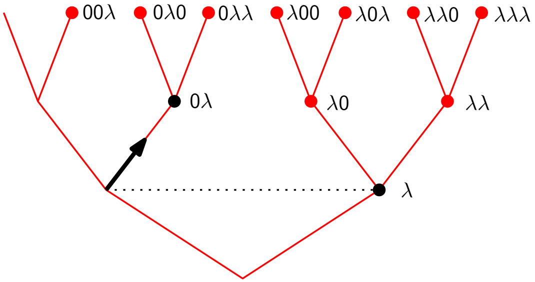

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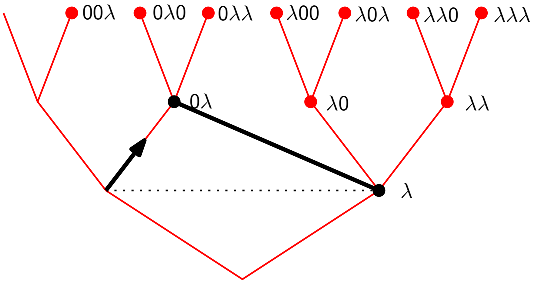

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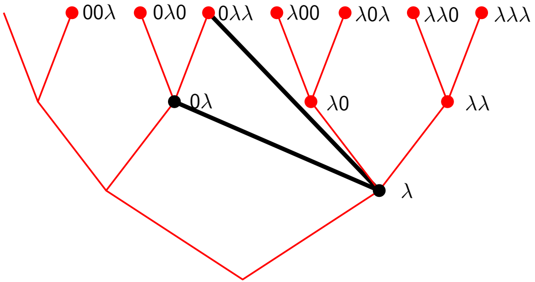

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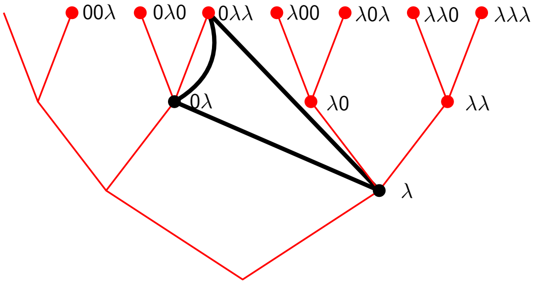

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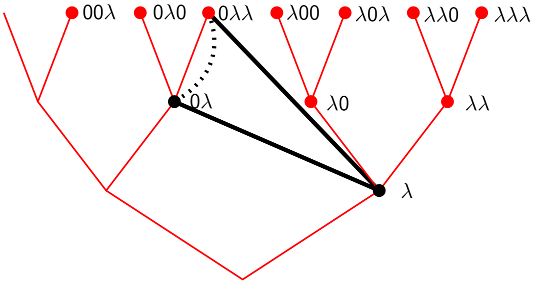

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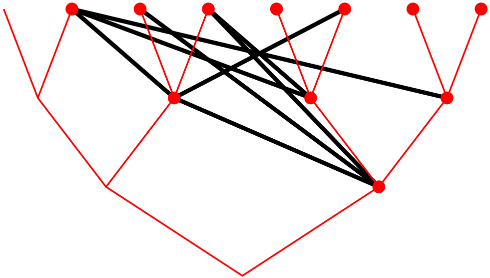

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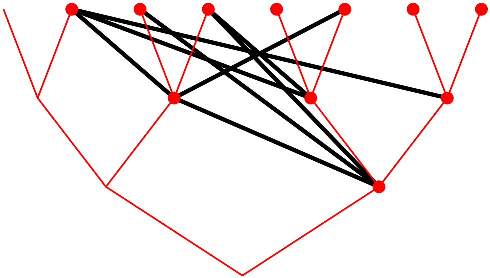

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Key observation 1: **G** is an universal triangle-free graph.

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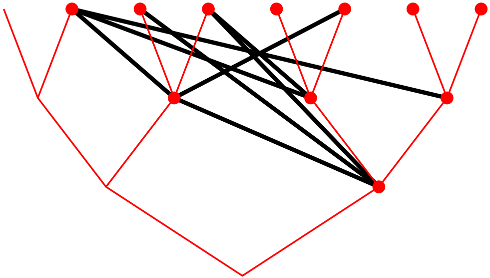

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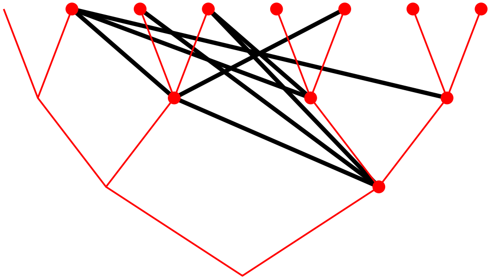
Key observation 1: **G** is an universal triangle-free graph.
Given any triangle-free graph H with vertex set ω assign every $i \in \omega$ word w of length i putting $\forall_{j<i}\, w_j = \lambda$ iff $\{i, j\}$ is an edge of H.

Triangle-free graph on 1-parameter words



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Key observation 1: **G** is an universal triangle-free graph.

Key observation 2: For every pair of 1-parameter words U and V and every ω-parameter W

$$U \sim V \iff W(U) \sim W(V).$$

Observation

$\mathbf{G}$ is a universal triangle-free graph.

Observation

For every infinite-parameter word W it holds that $u \sim v \iff W(u) \sim W(v)$.
(Substitution is also graph embedding on $\mathbf{G} \to \mathbf{G}$.)

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

Proposition (Envelopes are bounded)

There exists $T(|\Sigma|, s, k)$ such that for every set S of size s of k-parameter words in alphabet Σ there exists an envelope of S with at most $T(|\Sigma|, s, k)$ parameters.

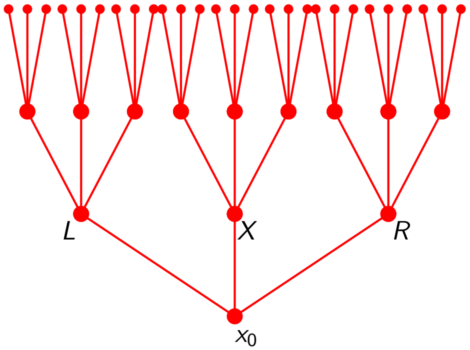
Theorem (Dobrinen 2020)

The big Ramsey degrees of universal triangle-free graph are finite.

Proof.

Fix graph $\mathbf{A}$ and a finite coloring of $\binom{\mathbf{G}}{\mathbf{A}}$. Because envelopes of copies of A are bounded, apply the theorem above for every embedding type and obtain a copy of $\mathbf{G}$ with bounded number of colors. $\square$

Partial order on infinite ternary tree



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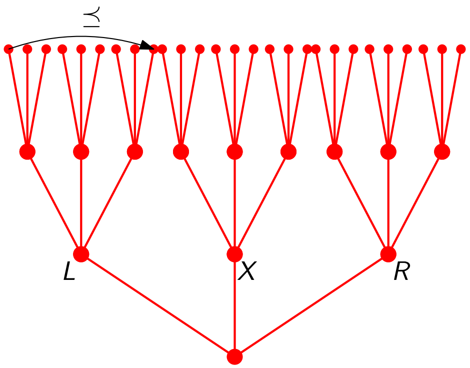


Put $\Sigma = \{L, X, R\}$ and order $L <_{\text{lex}} X <_{\text{lex}} R$.

Definition (Partial order $(\Sigma^*, \preceq)$)

For $w, w' \in \Sigma^*$ we put $w \prec w'$ if and only if there exists $0 \leq i < \min(|w|, |w'|)$ such that

1. $(w_i, w'_i) = (L, R)$ and
2. for every $0 \leq j < i$ it holds that $w_j \leq_{\text{lex}} w'_j$.

Key observations: $\preceq$ is universal partial order and is stable for substitution.

Big picture: proof techniques

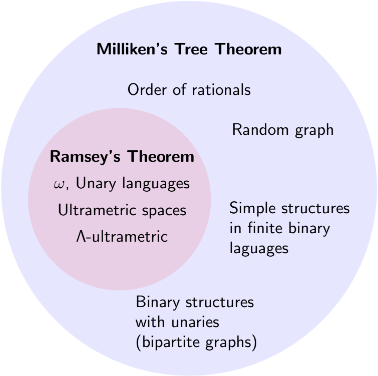
Ramsey's Theorem

ω, Unary languages

Ultrametric spaces

Λ-ultrametric

Big picture: proof techniques



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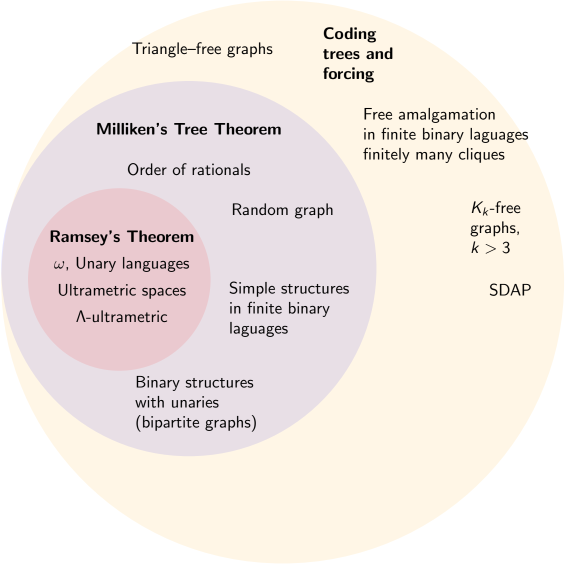

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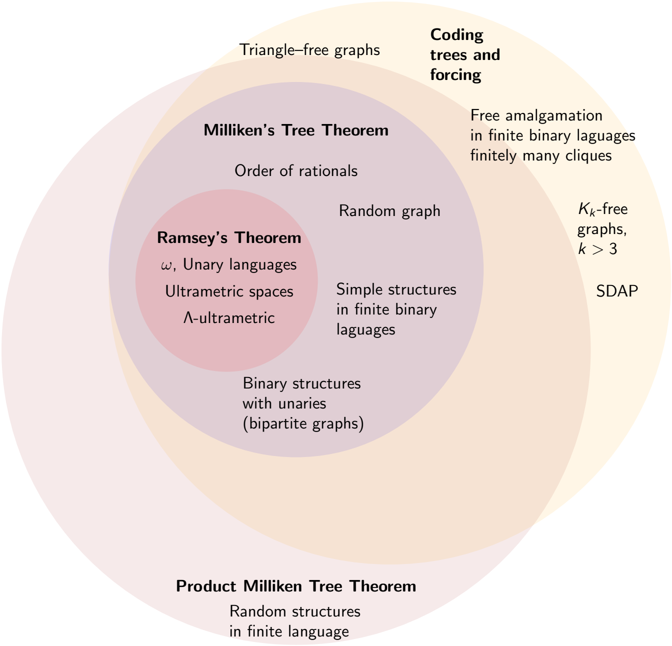

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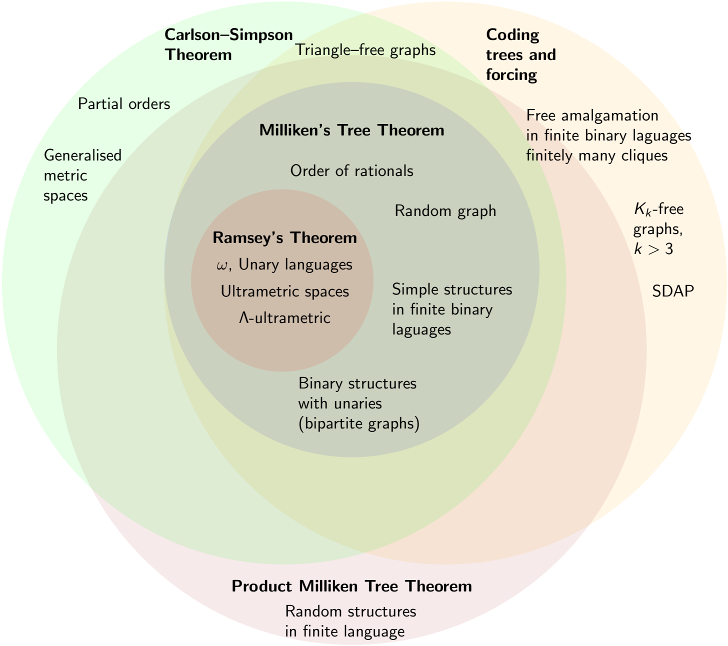
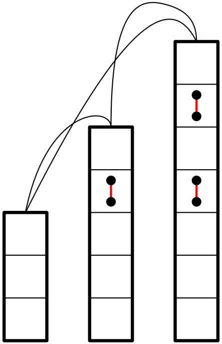


Why regular trees are good for K_3-free graphs but not good for K_4-free graphs?



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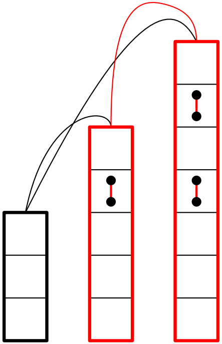

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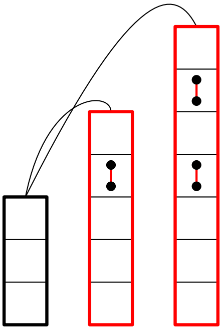

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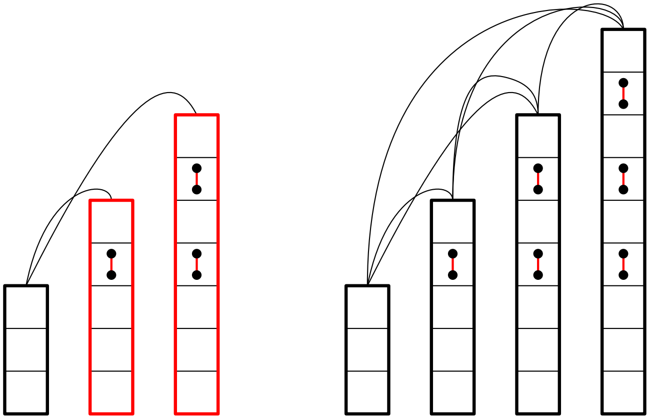

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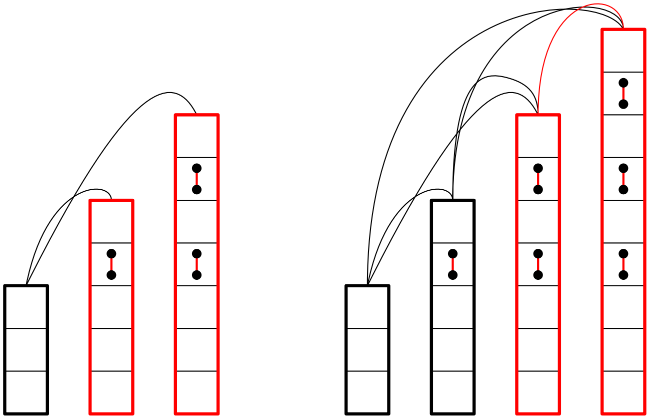
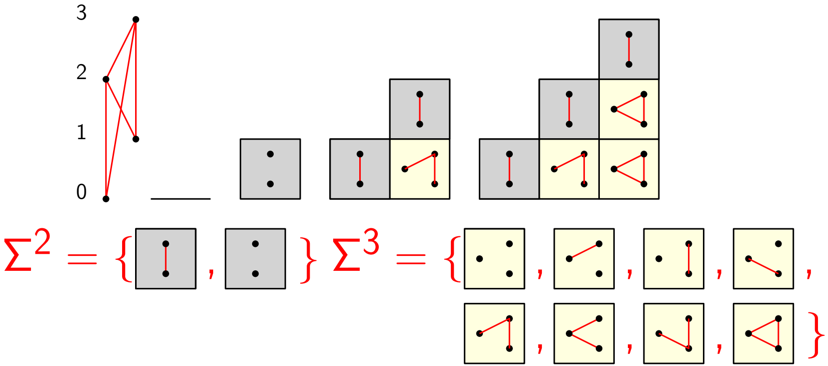
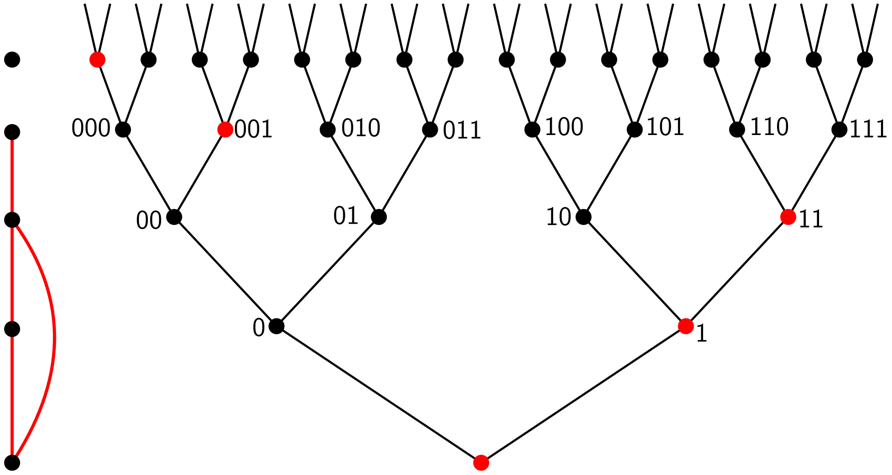


Higher order duals?



Coding tree (Dobrinen, Zucker)



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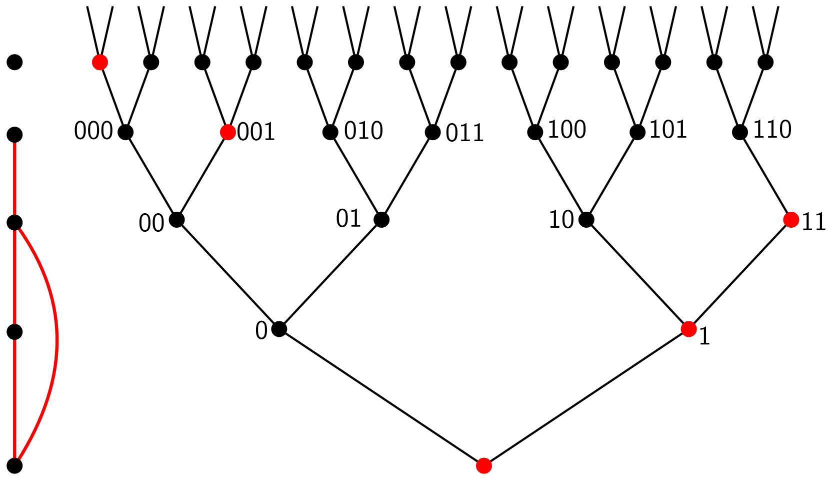
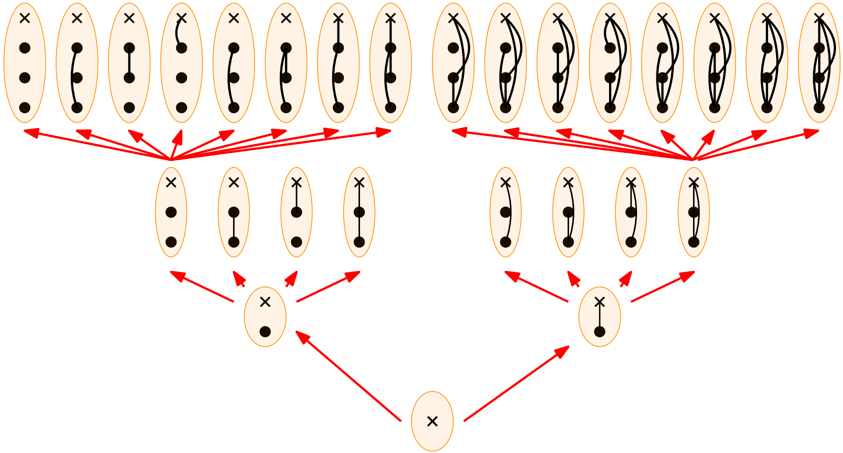


All enumerations tree



Basic idea: produce an amalgamation of all tree of types of enumerations of K_4-free graphs and make type remember the initial segment of enumeration it belongs to.

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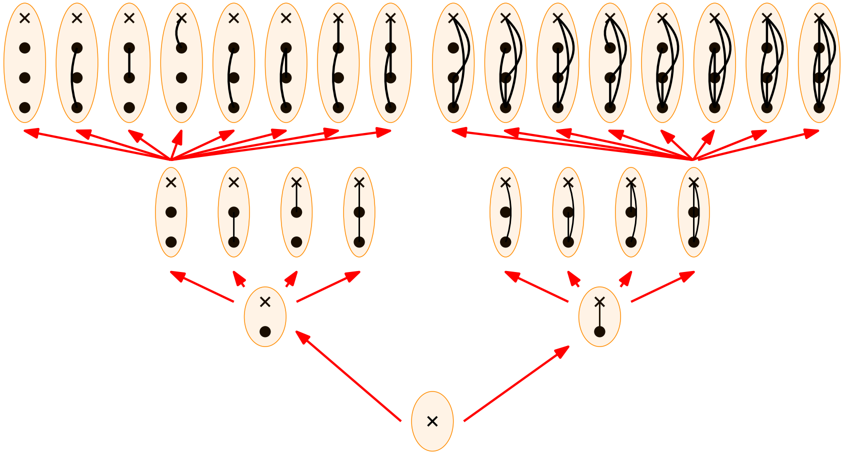

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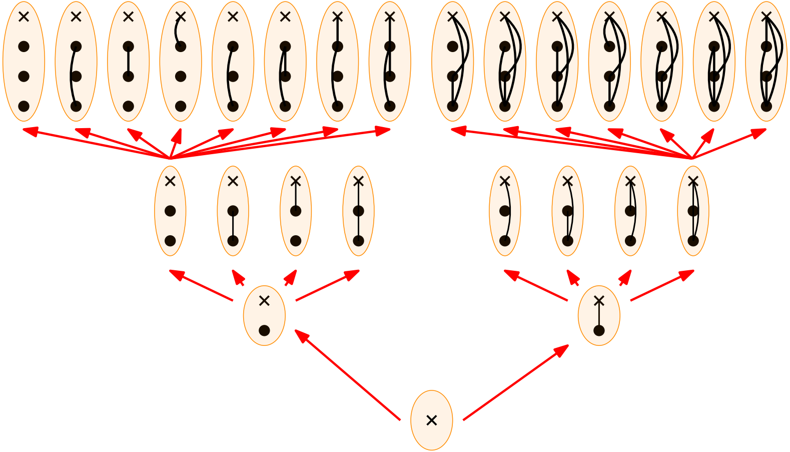

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Trees with a successor operation

While most Ramsey-type theorems are concerned about regularly branching trees, we need more general notion allowing trees with finite but unbounded branching.

Definition ($\mathcal{S}$-tree)

An $\mathcal{S}$-tree is a quadruple $(T, \preceq, \Sigma, \mathcal{S})$ where $(T, \preceq)$ is a countable finitely branching tree with finitely many nodes of level 0, Σ is a set called the alphabet and $\mathcal{S}$ is a partial function $\mathcal{S}\colon T \times T^{<\omega} \times \Sigma \to T$ called the successor operation satisfying the following three axioms:

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Example: a binary tree

Consider $\mathcal{S}$-tree is $(2^{<\omega}, \sqsubseteq, \{0, 1\}, \mathcal{S})$.
$\mathcal{S}$ is defined only for empty parameters $\bar{p}$ by concatenation: $\mathcal{S}(a, c) = a^\frown c$.

$$\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}((), 0), 1), 0), 1), 1) = 01011.$$

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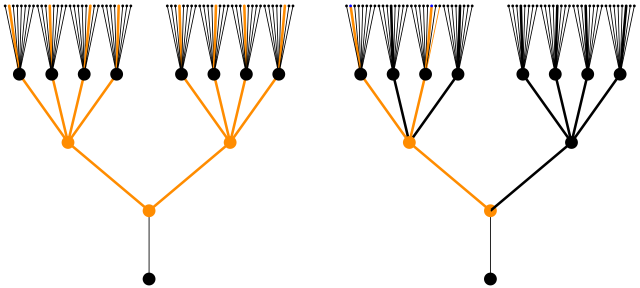

Shape-preserving functions

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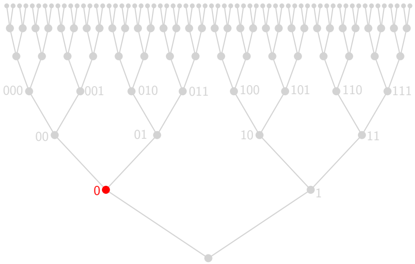

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Let $(T, \preceq, \Sigma, \mathcal{S})$ be an $\mathcal{S}$-tree. We call an injection $F\colon T \to T$ shape-preserving if

1. F is level preserving:
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3. For every $a \in T(0)$ and b such that $a \preceq b$ it also holds that $a \preceq F(b)$.

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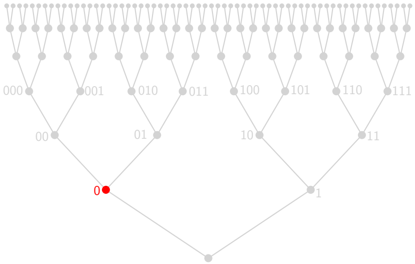

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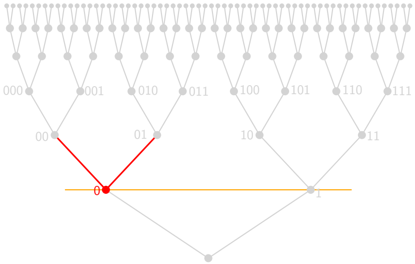

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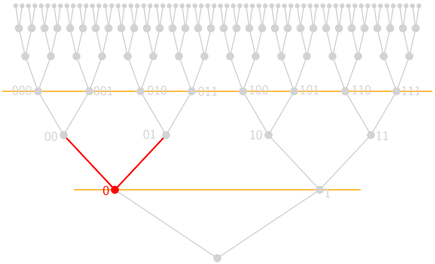

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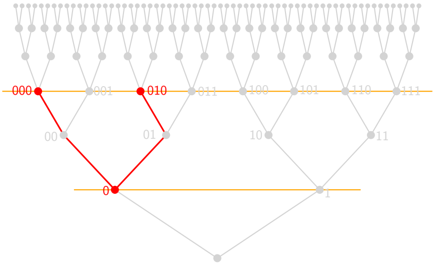

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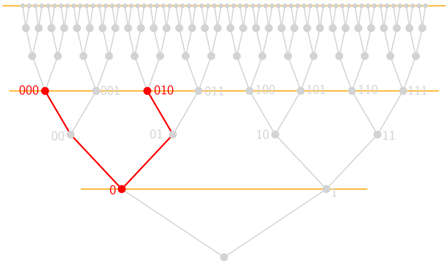

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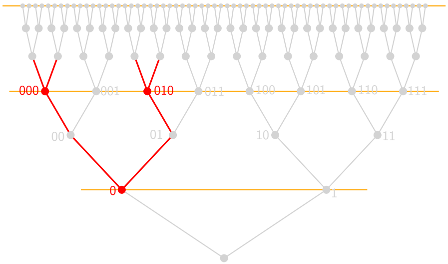

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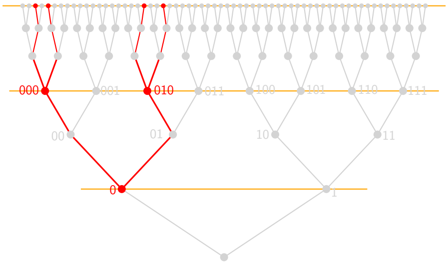

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For a level-preserving function $F\colon S \to T$, we denote by $\tilde{F}$ the function $\tilde{F}\colon \ell(S) \to \omega$ defined by $\tilde{F}(n) = \ell(F(a))$ for some $a \in S$ with $\ell(a) = n$.
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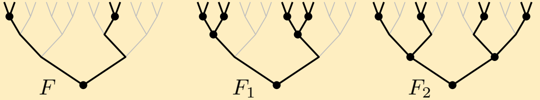

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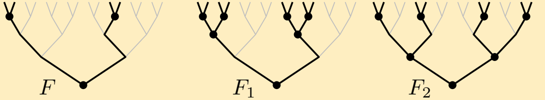
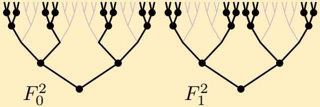


3. $\mathcal{M}$ is closed for duplication: For all n and m with $n < m \in \omega$, there exists a function $F^n_m \in \mathcal{M}$ skipping only level m such that for every $a \in T(n)$, $b \in T(m)$, $\bar{p} \in T^{<\omega}$ and $c \in \Sigma$, where $\mathcal{S}(a, \bar{p}, c)$ is defined and $\mathcal{S}(a, \bar{p}, c) \preceq b$, we have $F^n_m(b) = \mathcal{S}(b, \bar{p}, c)$.



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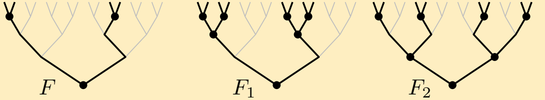

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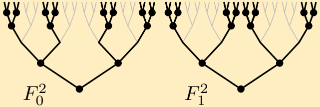

Ramsey theorem for trees with successor operation

Put $\mathcal{M}^n = \{F \in \mathcal{M} : F \restriction_{T(<n)} \text{ is identity}\}$, $\mathcal{AM}^n_k = \{F \restriction_{T(<n+k)} : F \in \mathcal{M}^n\}$.

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4. If $|\Sigma| > 1$ and $\mathcal{M}$ is generated only by duplication and "constant" functions we obtain Graham–Rothschild theorem theorem.

Ellentuck topology on $(\mathcal{S}, \mathcal{M})$-trees

Recall that a subset $\mathcal{X}$ of a topological space is

1. nowhere dense if every non-empty open set contains a non-empty open subset that avoids $\mathcal{X}$.
2. meager if is the union of countably many nowhere dense sets,
3. has the Baire property if it can be written as the symmetric difference of an open set and a meager set.

Put $\mathcal{AM} = \{F \restriction_{T(<n)} : F \in \mathcal{M}, n \in \omega\}$.

Definition (Ellentuck topological space $\mathcal{M}$)

Given an $(\mathcal{S}, \mathcal{M})$-tree $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ we equip $\mathcal{M}$ with the Ellentuck topology given by the following basic open sets:

$$[f, F] = \{F \circ F' : F' \in \mathcal{M} \text{ and } F \circ F' \text{ extends } f\}$$

for every $f \in \mathcal{AM}$ and $F \in \mathcal{M}$.

Topological Ramsey theorem for trees with successor operation

Given a shape-preserving function $F \in \mathcal{M}$ and $f\colon T(\leq n) \to T$ such that $f \in \mathcal{AM}$ we define $\mathrm{depth}_F(f) = \tilde{g}(n)$ for $g \in \mathcal{AM}$ satisfying $F \circ g = f$. We set $\mathrm{depth}_F(f) = \omega$ if there is no such g.

Definition

Let $\mathcal{X}$ be a subset of $\mathcal{M}$.

1. We call $\mathcal{X}$ Ramsey if for every non-empty basic set $[f, F]$ there is $F' \in [F \restriction_{\mathrm{depth}_F} (f), F]$ such that either $[f, F'] \subseteq \mathcal{X}$ or $[f, F'] \cap \mathcal{X} = \emptyset$.
2. We call $\mathcal{X}$ Ramsey null if for every $[f, F] \neq \emptyset$ we can find $F' \in [F \restriction_{\mathrm{depth}_F(f)}, F]$ s. t. $[f, F'] \cap \mathcal{X} = \emptyset$.

Theorem (Ellentuck theorem for shape-preserving functions)

Let $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ be an $(\mathcal{S}, \mathcal{M})$-tree and consider $\mathcal{M}$ with the Ellentuck topology. Then every property of Baire subset of $\mathcal{M}$ is Ramsey and every meager subset is Ramsey null.

Examples

Consider $\mathcal{S}$-tree $(\Sigma^{<\omega}, \sqsubseteq, \Sigma, \mathcal{S})$ for some finite alphabet Σ.

1. If $|\Sigma| = 0$ we obtain Ellentuck theorem.

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3. If $|\Sigma| > 1$ and $\mathcal{M}$ is generated only by duplication functions $\implies$ Carlson–Simpson theorem.

Proof outline

1. 1-dimensional pigeonhole is proved using Hales-Jewett theorem (duplication is important here).
2. Method of (combinatorial) forcing is used to prove ω-dimensional pigeonhole on a stronger notion of “fat subtrees”.
3. Todorčević axioms of Ramsey spaces are used to obtain a Ramsey space of fat subtrees.
4. Topological Ramsey theorem for trees with successor operation follows as a consequence.

We obtain an interesting example of Ramsey space where Todorčević A3.2 axiom is not satisfied.

Our new theorem on regularly branching trees

Definition (Boring extensions)

Given finite alphabet Σ, a family of boring extensions is a countable sequence $\mathcal{E} = (\mathcal{E}_n)_{n\in\omega}$ of sets

$$\mathcal{E}_n \subseteq \{e \text{ is a function } e\colon \Sigma^n \to \Sigma\}$$

satisfying the following two conditions:

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1. Duplication: For every $m < n$ the set $\mathcal{E}_n$ contains a function $e^n_m\colon \Sigma^n \to \Sigma$ defined by:

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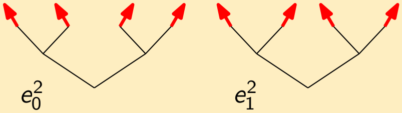

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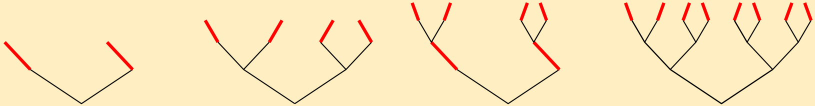
2. Insertion: For every $m \leq n$, $e_1 \in \mathcal{E}_m$, $e_2 \in \mathcal{E}_n$ there exists $e_3 \in \mathcal{E}_{n+1}$ such that for every $a \in \Sigma^m$ and $b \in \Sigma^{n-m}$ the following is satisfied:

$$e_3(a^\frown e_1(a)^\frown b) = e_2(a^\frown b).$$



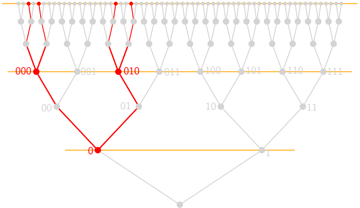
e_1 e_2 "inserting" e_1 to e_2 e_3

Generalized embedding types

Definition (Interesting levels)

Given a finite alphabet Σ, a family of ~~boring~~ extensions $\mathcal{E}$ and a set $X \subseteq \Sigma^{<\omega}$ we denote by $I_{\mathcal{E}}(X)$ the set of interesting levels of X. This is a set of all $\ell \in \omega$ such that there is no $e_\ell \in \mathcal{E}_\ell$ satisfying for every $a \in X$, $|a| \geq \ell$

$$(a|_\ell)^\frown e_\ell(a|_\ell) \sqsubseteq a.$$



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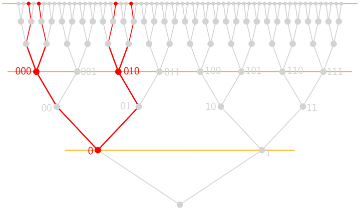

Definition (Embedding type)

Given a finite alphabet Σ, a family of ~~boring~~ extensions $\mathcal{E}$ and a set $X \subseteq \Sigma^{<\omega}$ we define the embedding type of X, denoted $\tau_{\mathcal{E}}(X)$, to be the set of all words created from words in X by removing all letters with indices not in $I_{\mathcal{E}}(X)$.

Coloring subsets of a given embedding type

Given a finite alphabet Σ, a family of ~~boring~~ extensions $\mathcal{E}$ and sets $X, Y \subseteq \Sigma^{<\omega}$ we put

$$\binom{Y}{X/\mathcal{E}} = \{X' \subseteq Y : \tau_{\mathcal{E}}(X') = \tau_{\mathcal{E}}(X)\}.$$

Theorem (BCDHKNZ 2023+, Colouring sets of given embedding type in a finite tree)

For every finite alphabet Σ, family of ~~boring~~ extensions $\mathcal{E}$, positive integer r, finite set $X \subseteq \Sigma^{<\omega}$ and (possibly infinite) set $Y \subseteq \Sigma^{<\omega}$ and every finite colouring χ of $\binom{\Sigma^{<\omega}}{X/\mathcal{E}} \to r$ there exists $Y' \in \binom{\Sigma^{<\omega}}{Y/\mathcal{E}}$ such that χ is constant when restricted to $\binom{Y'}{X/\mathcal{E}}$.

Theorem (BCDHKNZ 2023+, Colouring sets of given embedding type in a combinatorial cube)

For every finite alphabet Σ, family of ~~boring~~ extensions $\mathcal{E}$, positive integers m, n, r and (finite) sets $X \subseteq \Sigma^m$, $Y \subseteq \Sigma^n$ there exists $N \in \omega$ such that for every r-colouring $\chi\colon \binom{\Sigma^N}{X/\mathcal{E}} \to r$ there exists $Y' \in \binom{\Sigma^N}{Y/\mathcal{E}}$ such that χ is constant when restricted to $\binom{Y'}{X/\mathcal{E}}$.

Thank you for the attention

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