Higher order dualization of the Ramsey theorem and big Ramsey degrees

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Joint work with Martin Balko, Natasha Dobrinen, David Chodounský, Matěj Konečný, Jaroslav Nešetřil, Stevo Todorcevic, Lluis Vena, Andy Zucker

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Big Ramsey degrees using parameter spaces

In Matěj's talk we learnt how to apply parameter spaces to give upper bound on big Ramsey degrees:

Example

- Free amalgamation classes in binary language with forbidden triangles.
- Metric spaces
- Ultrametric spaces
- Partial orders
- . . .

(In general method works well for strong amalgamation classes with triangle constraints)

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Today talk

Can we generalize the method to the homogeneous universal K_4-free graph $\mathbf{R}_4$?

Big Ramsey degrees are known to be finite due to Dobrinen (2019+) with simplified proof by Zucker (2020+).

Gluing enumerations together

Given structure $\mathbf{H}$ we say that structure $\mathbf{H}_1$ is an enumeration of $\mathbf{H}$ iff it is isomorphic to $\mathbf{H}$ and its domain is $|H|$.

Definition (Structure $\mathbf{G_H}$)

Let $\mathbf{H}$ be a countably-infinite homogeneous structure. Then we denote by $\mathbf{G_H}$ the structure created from the disjoint union of all enumerations of $\mathbf{H}$ by identifying vertex i of $\mathbf{H}_1$ with vertex i of $\mathbf{H}_2$ if and only if it holds that for every $j < i$ the structure induced by $\mathbf{H}_1$ on $\{i, j\}$ is same as the structure induced by $\mathbf{H}_2$ on $\{i, j\}$.

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This construction applied on $\mathbf{R}_4$ produces K_4

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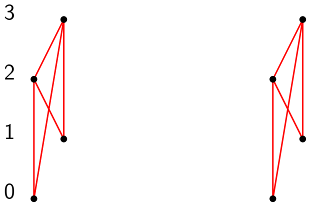

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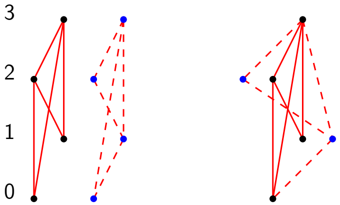

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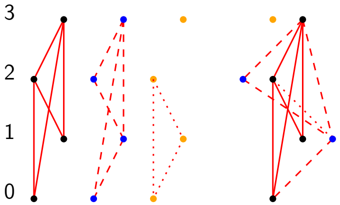

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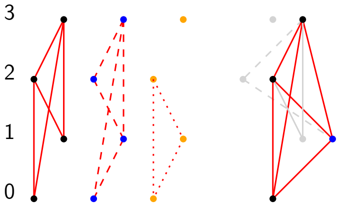

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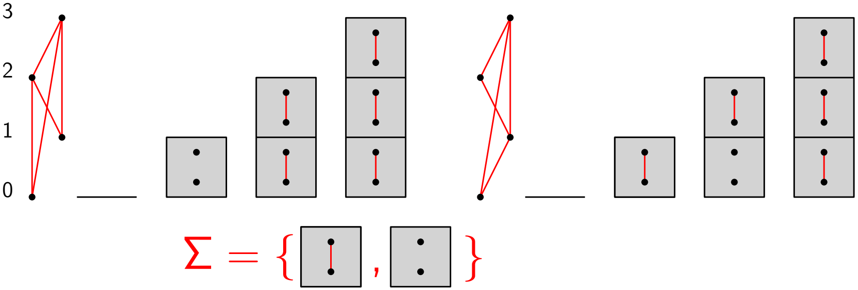

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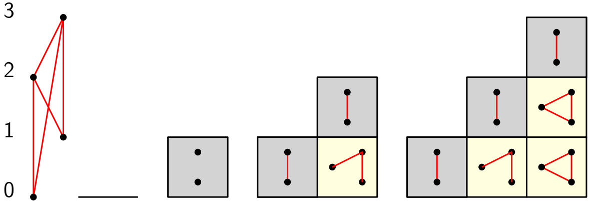
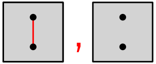
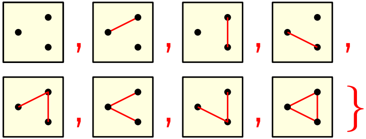


$\Sigma^2 = \{$  $\}$ $\Sigma^3 = \{$  $\}$

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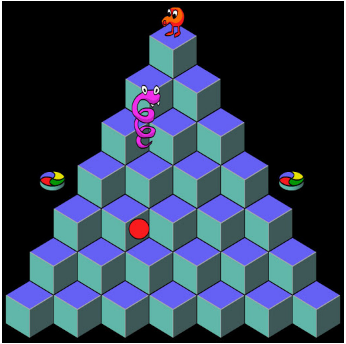

Ramsey theory on Qberts

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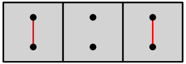
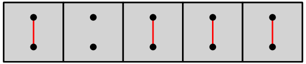
We can apply parameter words on our vertices if we allow some entries to be undefined (gaps)

$\lambda_0\lambda_1\lambda_0\lambda_2\lambda_2($  $) =$ 

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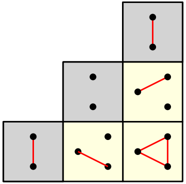
$\lambda_0\lambda_1\lambda_0\lambda_2\lambda_2($  $) =$

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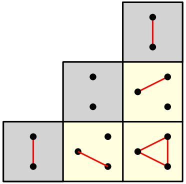

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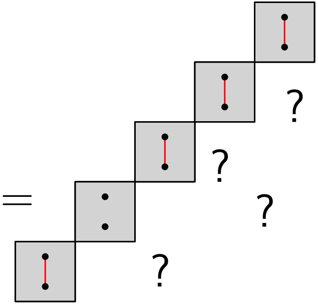

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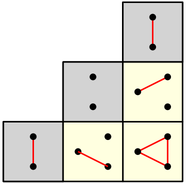
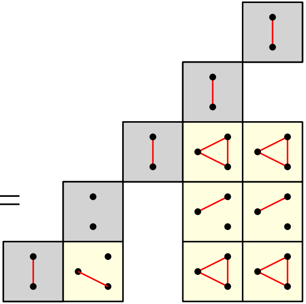
$\lambda_0\lambda_1\lambda_0\lambda_2\lambda_2($  $) =$ 

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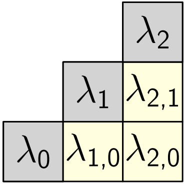

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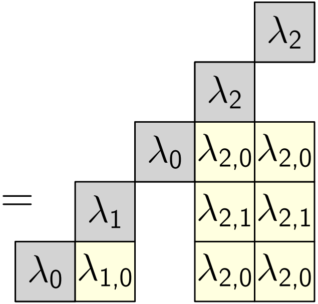

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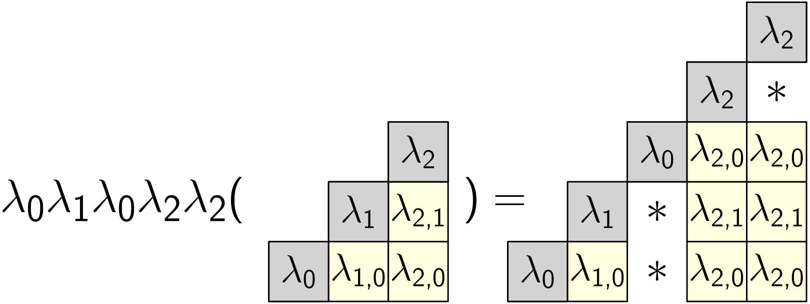

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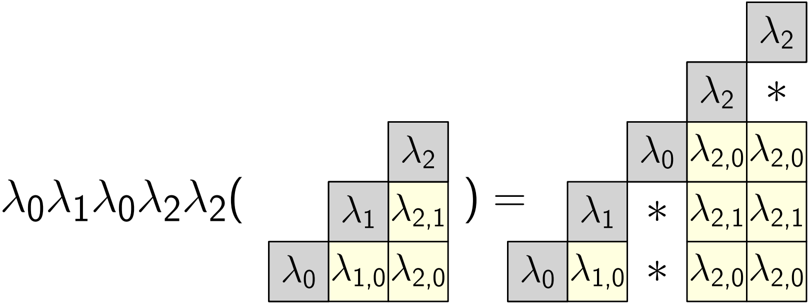

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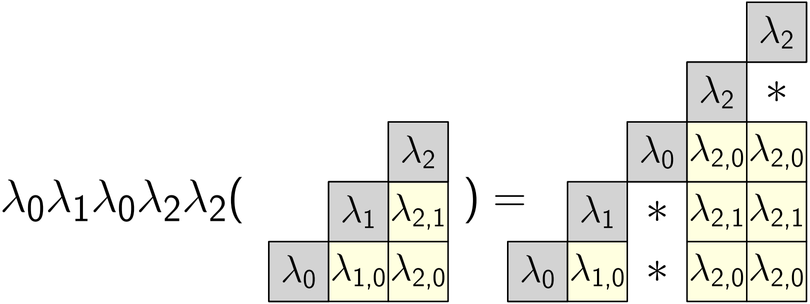

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Bad news

The family of embeddings $\overline{\mathbf{G}}^3_{\mathbf{H}} \to \overline{\mathbf{G}}^3_{\mathbf{H}}$ corresponding to substitutions is not rich enough to make envelopes finite.

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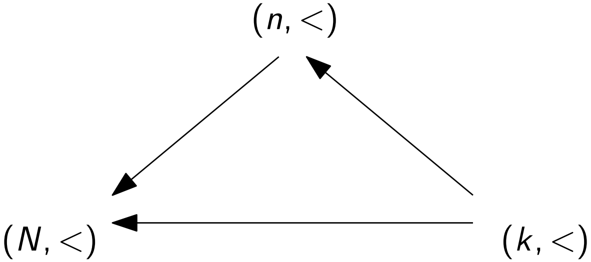

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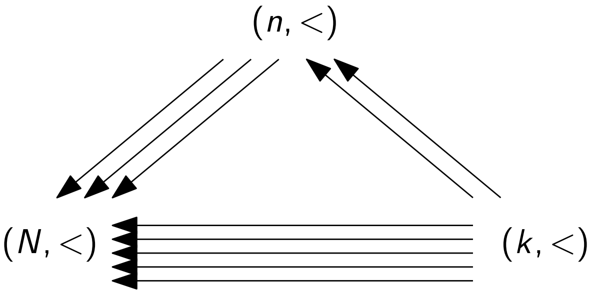

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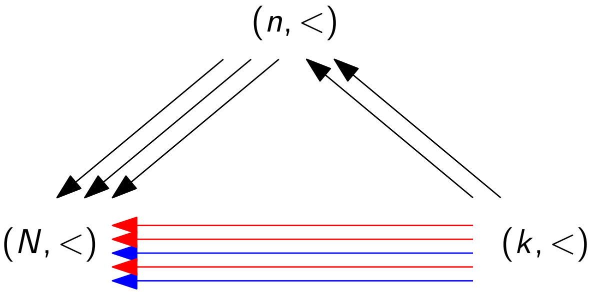

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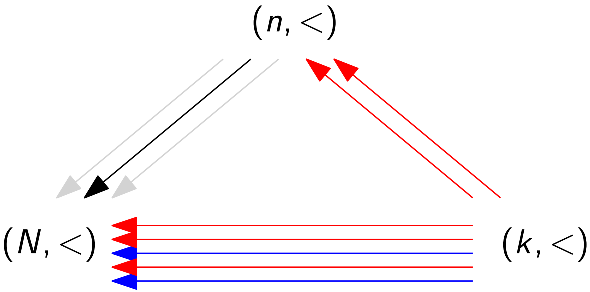

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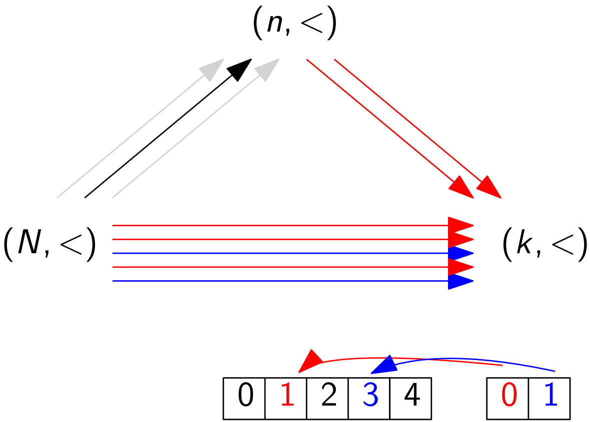

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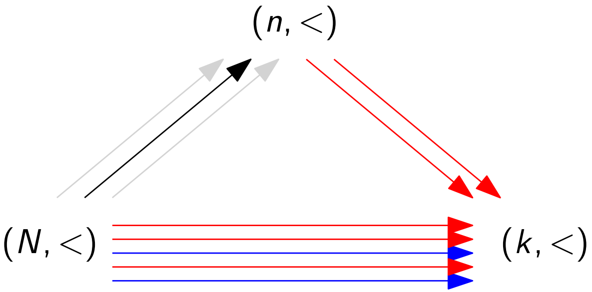

0	1	1	2	2

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Relaxation

It is possible to relax monotonous surjection to rigid surjections.

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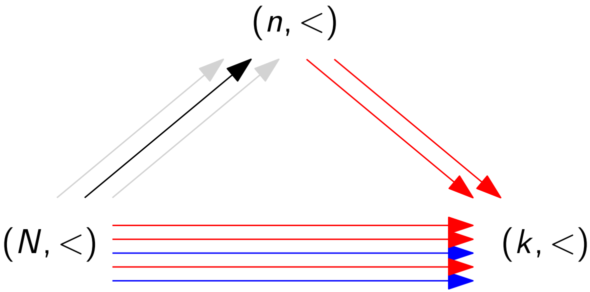

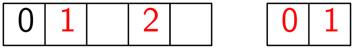

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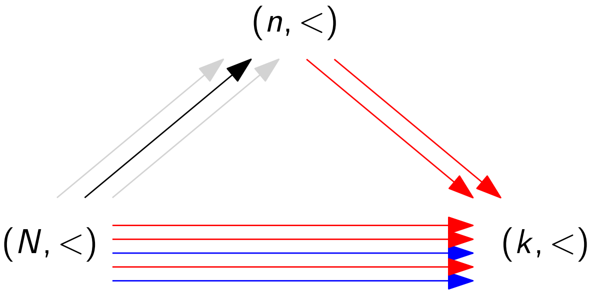

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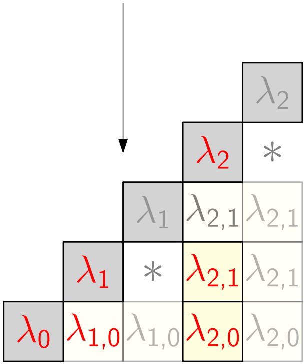

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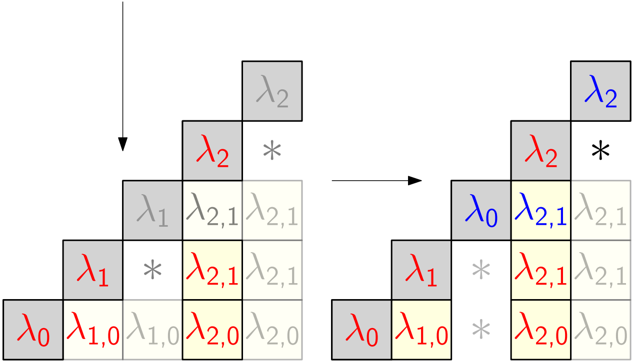

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2. Turn monotonous surjection on diagonal to rigid surjection: preserve first occurrences and adjust other entries.

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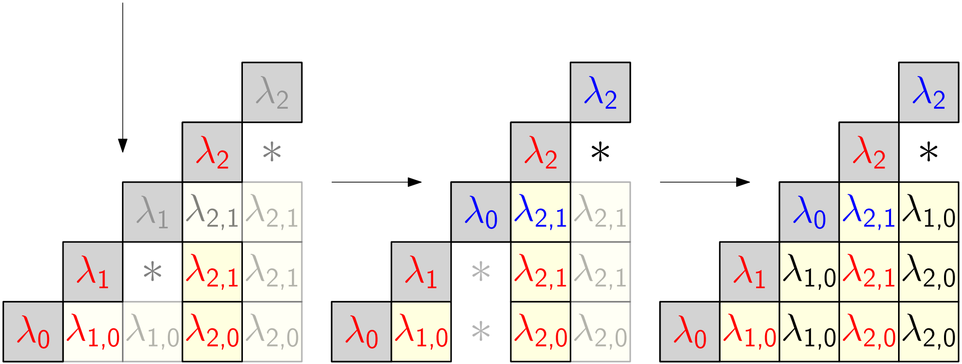

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2. Turn monotonous surjection on diagonal to rigid surjection: preserve first occurrences and adjust other entries.
3. Keep columns corresponding to first occurrecnes as given by the diagonal. Relax other entries while preserving fist occurrences.

Higher order dual Ramsey theorem: Index sets

Index sets

Given a positive integer o and $n \in \omega + 1$, we denote by I^o_n the set of all vectors $(i_0, i_1, \ldots, i_{\ell-1})$ with $0 < \ell \leq o$ such that $0 \leq i_0, i_\ell < n$, and $i_j + 2 \leq i_{j+1}$ for every $0 \leq j < \ell - 1$. The index set of order o is the set I^o_ω which we also denote by I^o.

Example

$I^1 = \{(0), (1), (2), \ldots\}$. I^2 can be visualized as follows:

$$\begin{pmatrix}
 & & & & & & \iddots \\
 & & & & & (5) & \\
 & & & & (4) & & \iddots \\
 & & & (3) & & (3,5) & \cdots \\
 & & (2) & & (2,4) & (2,5) & \cdots \\
 & (1) & & (1,3) & (1,4) & (1,5) & \cdots \\
(0) & & (0,2) & (0,3) & (0,4) & (0,5) & \cdots
\end{pmatrix}$$

Higher order dual Ramsey theorem: Words

Given a set R not containing $*$, integers $o \geq 2$ and $i \geq 0$, and a function $W\colon I^o \to R \cup \{*\}$, the ith slice of W is a function $W[i]\colon I^{o-1} \to R \cup \{*\}$ defined by setting, for every $\vec{j} \in I^{o-1}$,

$$W[i](\vec{j}) = \begin{cases} W(\vec{j}^\frown i) & \text{if } \vec{j}^\frown i \in I^o, \\ * & \text{otherwise.} \end{cases}$$

Definition (Word)

Given a finite alphabet Σ not containing $*$ and a positive integer o, we define words of order o inductively in the following way. A function $W\colon I^o \to \Sigma \cup \{*\}$ is a word of order o in the alphabet Σ if W satisfies the following conditions:

1. For all integers i and j such that $j \geq i \geq 0$ and $W(i) = *$, we have $W(j) = *$.
2. If $o > 1$, then the following two conditions are satisfied for every $i \geq 2$:
    1. If $W(i) = *$, then $W(0, i) = *$, and
    2. the ith slice $W[i]$ is a word of order $o - 1$ in the alphabet Σ.

Higher order dual Ramsey theorem: Parameter words

Example

$$E^6 = \begin{pmatrix} & & & & & \lambda_5 \\ & & & & \lambda_4 & \\ & & & \lambda_3 & & \lambda_{(3,5)} \\ & & \lambda_2 & & \lambda_{(2,4)} & \lambda_{(2,5)} \\ & \lambda_1 & & \lambda_{(1,3)} & \lambda_{(1,4)} & \lambda_{(1,5)} \\ \lambda_0 & & \lambda_{(0,2)} & \lambda_{(0,3)} & \lambda_{(0,4)} & \lambda_{(0,5)} \end{pmatrix}$$

Example

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Higher order dual Ramsey theorem: Parameter words

Definition (Parameter word of order one)

Given an alphabet Σ and $k \in \omega + 1$, a k-parameter word of order 1 in the alphabet Σ is a word W of order 1 in the alphabet $\Sigma \cup \{\lambda_i : 0 \leq i < k\}$ such that, for every $0 \leq i \leq k$, the first occurrence $|W|_{\lambda_i}$ of λ_i is finite and, if $i > 0$, we also have $|W|_{\lambda_{i-1}} < |W|_{\lambda_i}$.

The diagonal subword W^D of W is a function $W^D\colon I^{o-1} \to \Sigma \cup \{*\}$ with entries $W^D(\vec{j}) = W_{\vec{j}}$ for every $\vec{j} \in I^{o-1}$. Given $c \in \Sigma$, the first occurrence of c in W, denoted by $|W|_c$, is the minimum integer $i \geq 0$ satisfying $W_i = c$ or ω if there is no such i. We abbreviate $|W|_*$ by $|W|$ and call it the length of the word W.

Definition (Parameter word of order $o \geq 2$)

Given an alphabet Σ, an integer $o \geq 2$, and $k \in \omega + 1$, a k-parameter word of order o in the alphabet Σ is a word W of order o in the alphabet $\Sigma \cup \{\lambda_{\vec{i}} : \vec{i} \in I^o_k\}$ satisfying the following conditions:

1. W^D is a k-parameter word of order $o - 1$ in the alphabet Σ and
2. $W[|W|_{\lambda_0}]_{\vec{j}} = *$ for every $\vec{j} \in I^{o-1}_{|W|_{\lambda_0}-1}$.
3. For all integers p, $1 \leq p < k$, for $i = |W|_{\lambda_p}$, and every $\vec{j} \in I^{o-1}_{i-1}$ we have

$$W[i]_{\vec{j}} = \begin{cases} \lambda_{\vec{q}^\frown p} & \text{if } \vec{j} \in I^{o-1}_{|W|_{\lambda_{p-1}}} \text{ and } W_{\vec{j}} = \lambda_{\vec{q}} \text{ for some } \vec{q} \in I^{o-1}_{p-1}, \\ W_{\vec{j}} & \text{if } \vec{j} \in I^{o-1}_{|W|_{\lambda_{p-1}}} \text{ and } W_{\vec{j}} \in \Sigma, \\ * & \text{otherwise.} \end{cases}$$

4. If $W_{\vec{j}} = \lambda_{\vec{p}}$ for some $\vec{j}, \vec{p} \in I^o$, then $|\vec{p}| = |\vec{j}|$ and $\vec{j} \notin I^o_{|W|_{\lambda_{\max \vec{p}}}}$.

Higher order dual Ramsey theorem: Substitution

Example

$$U = \begin{pmatrix} & \lambda_0 \\ 0 & \end{pmatrix} \qquad W = \begin{pmatrix}
 & & & & & & & \lambda_1 \\
 & & & & & & \lambda_2 & \\
 & & & & & \lambda_2 & & 1 \\
 & & & & \lambda_1 & & * & 0 \\
 & & & 1 & & 1 & * & \lambda_{(0,2)} \\
 & & \lambda_0 & & * & \lambda_{(0,2)} & * & 1 \\
 & 0 & & 1 & 0 & 0 & * & \lambda_{(0,2)} \\
1 & & * & 0 & 1 & 1 & \lambda_{(0,2)} & 0
\end{pmatrix}$$

$$W(U) =$$

Higher order dual Ramsey theorem: Substitution

Example

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$$W(U) = \begin{pmatrix} & & & & & & & \lambda_0 \\ & & & & & & * & \\ & & & & & * & & 1 \\ & & & & \lambda_0 & & * & 0 \\ & & & 1 & & 1 & * & * \\ & & 0 & & * & * & * & 1 \\ & 0 & & 1 & 0 & 0 & * & * \\ 1 & & * & 0 & 1 & 1 & * & 1 \end{pmatrix}$$

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$$W(U) = \begin{pmatrix} & & & & & & & {\color{red}\ast} \\ & & & & & & \ast & \\ & & & & & \ast & & 1 \\ & & & & \lambda_0 & & \ast & 0 \\ & & & 1 & & 1 & \ast & \ast \\ & & 0 & & \ast & \ast & \ast & 1 \\ & 0 & & 1 & 0 & 0 & \ast & \ast \\ 1 & & \ast & 0 & 1 & 1 & \ast & 1 \end{pmatrix}$$

Higher order dual Ramsey theorem: Substitution

Example

$$U = \begin{pmatrix} & \lambda_0 \\ 0 & \end{pmatrix} \qquad W = \begin{pmatrix}
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 & & & & & & \lambda_2 & \\
 & & & & & \lambda_2 & & 1 \\
 & & & & \lambda_1 & & \ast & 0 \\
 & & & 1 & & 1 & \ast & \lambda_{(0,2)} \\
 & & \lambda_0 & & \ast & \lambda_{(0,2)} & \ast & 1 \\
 & 0 & & 1 & 0 & 0 & \ast & \lambda_{(0,2)} \\
1 & & \ast & 0 & 1 & 1 & \lambda_{(0,2)} & 0
\end{pmatrix}$$

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 & & & & & & & {\color{red}\ast} \\
 & & & & & & \ast & \\
 & & & & & \ast & & {\color{red}\ast} \\
 & & & & \lambda_0 & & \ast & {\color{red}\ast} \\
 & & & 1 & & {\color{red}\ast} & \ast & \ast \\
 & & 0 & & \ast & \ast & \ast & {\color{red}\ast} \\
 & 0 & & 1 & 0 & {\color{red}\ast} & \ast & \ast \\
1 & & \ast & 0 & 1 & {\color{red}\ast} & \ast & {\color{red}\ast}
\end{pmatrix}$$

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Higher order dual Ramsey theorem: Substitution

Definition (Pruning)

Given a finite alphabet Σ, a positive integer o, and a function $W \colon I^o \to \Sigma \cup \{*\}$, we let $p(W) \colon I^o \to \Sigma \cup \{*\}$ be the function such that:

1. For every $i \geq 0$,

$$p(W)_i = \begin{cases} W(i) & \text{if } i \leq |W| \\ * & \text{otherwise.} \end{cases}$$

2. For every $i \geq 0$ and $\vec{j} \in I_{i-1}^{o-1}$

$$p(W[i])_{\vec{j}} = \begin{cases} p(W[i]_{\vec{j}}) & \text{if } 1 < i < |W|, i \neq |W|_{\lambda_0} \\ * & \text{otherwise} \end{cases}$$

Definition (Substitution)

Let Σ be a finite alphabet not containing $*$, o be a positive integer, $k \in \omega + 1$, and let W be a k-parameter word of order o in the alphabet Σ. Consider a parameter word U of order o and of length at most k in the alphabet Σ. The substitution of U to W produces a parameter word $W(U)$, which is defined as $p(W')$ where $W' \colon I^o \to \Sigma \cup \{*\}$ is a function defined by setting

$$W'_{\vec{i}} = \begin{cases} W_{\vec{i}} & \text{if } W_{\vec{i}} \in \Sigma, \\ U_{\vec{p}} & \text{if } W_{\vec{i}} = \lambda_{\vec{p}} \text{ for some } \vec{p} \in I_k^o. \\ * & \text{otherwise.} \end{cases}$$

for every $\vec{i} \in I^o$.

Higher order dual Ramsey theorem: Statement

1. For $k \leq n \in \omega + 1$, let
$$[\Sigma]_o \binom{n}{k}$$
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3. Given a k-parameter word $W \in [\Sigma]_o \binom{n}{k}$ and a set $S \subseteq \bigcup_{\ell \leq k} [\Sigma]_o^* \binom{k}{\ell}$, we put
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Theorem (Balko, Chodounský, H., Konečný, Vena, 2020+)

Let Σ be a finite alphabet, o be a positive integer, and $k \in \omega$. If the set $[\Sigma]^*_o \binom{\omega}{k}$ is coloured by finitely many colours, then there exists $W \in [\Sigma]_o \binom{\omega}{\omega}$ such that $W\left([\Sigma]^*_o \binom{\omega}{k}\right)$ is monochromatic.

For $o = 1$ this is known as Voight (or Carlson-Simpson) Lemma.

Corollaries

Theorem (Zucker 2020)

Fraïssé limits of free amalgamation classes in finite binary language with finitely many constraints have finite big Ramsey degrees.

Theorem (Balko, Chodounský, H., Nešetřil, Vena, 2019+; Coulson, Dobrinen, Patel 2020+)

Fraïssé limits of unrestricted amalgamation classes in finite relational language.

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Theorem (Balko, Chodounský, H., Konečný, Nešetřil, Vena, 2020+)

Metrically homogeneous graphs of finite diameter from Cherlin's catalogue have finite big Ramsey degrees.

Big Ramsey degrees for some structures in non-binary languages.
It is possible to define Ramsey space similar to one given by Carlson-Simpson. This is joint work in progress with Stevo Todrocevic.

Proof outline for free amalgamation classes in binary language

Let **H** be a binary free amalgamation structure, $c > 1$. Denote by $\mathcal{K}$ the age of **H**. For simplicity assume that all substructures of size 1 are isomorphic. Let Σ be the class of all enumerated structures in $\mathcal{K}$ with at most c vertices.

Definition (Structure $\overline{\mathbf{G}}^c_{\mathbf{H}}$)

1. Vertex set $\overline{\mathbf{G}}^c_{\mathbf{H}}$ consists of all words $U \in [\Sigma]^*_{c-1}\binom{\omega}{0}$ such that:
   1. $|U| > 0$.
   2. For every $\vec{i} \in I^{c-1}$ such that $U_i \neq *$ it holds that $|U_i| = |\vec{i}| - 1$.

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2. For $R \in L$ of arity 2 and vertices $U^0, U^1 \in G_{\mathbf{H}^c}$ we put $(U^0, U^1) \in R_{\overline{\mathbf{G}}_{\mathbf{H}^c}}$ if the following is satisfied:
   1. Presence: Either $|U^1| < |U^0|$ and $(0, 1) \in R_{\mathbf{U}^0_{|U^1|}}$ or $|U^0| < |U^1|$ and $(1, 0) \in R_{\mathbf{U}^1_{|U^0|}}$.

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   2. Diagonal projection: Denote by U the shorter word in $\{U^0, U^1\}$ and by U' the longer. For every $\vec{i} \in I^{c-2}_{|U|-1}$ it holds that

      $$(U_{\vec{i}} = *) \implies (U'[|U|]_{\vec{i}} = *)$$

      and if both are not $*$ then
      1. $\mathbf{U}_{\vec{i}}$ is created from structure $\mathbf{U}'[|U|]_{\vec{i}}$ by removing maximal vertex and
      2. $\mathbf{U}'[|U|]_{\vec{i}}$ extends $\mathbf{U}'_{\vec{i}}$ by next-to-last vertex.

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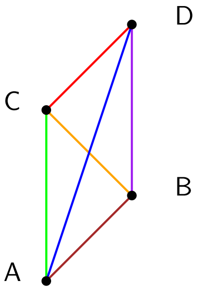
      2. $\mathbf{U}'[|U|]_{\vec{i}}$ extends $\mathbf{U}'_{\vec{i}}$ by next-to-last vertex.
   3. Slice consistency: For every $\vec{i} \in I^{c-2}_{|U|-1}$ it holds that
   $$(U'[|U|]_{\vec{i}} \neq *) \implies (U'_{\vec{i}} \neq *)$$
   and if both are not $*$ then mapping $0 \mapsto |\vec{i}|$, $1 \mapsto |\vec{i}| + 1$ is an embedding of $\mathbf{U}'_{|U|} \to \mathbf{U}'[|U|]_{\vec{i}}$.

$\mathrm{Age}(\overline{\mathbf{G}}^{c}_{\mathbf{H}}) = \mathrm{Age}(\mathbf{H})$



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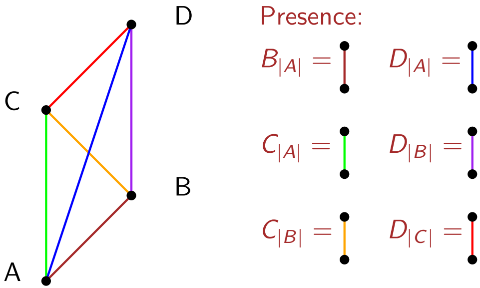
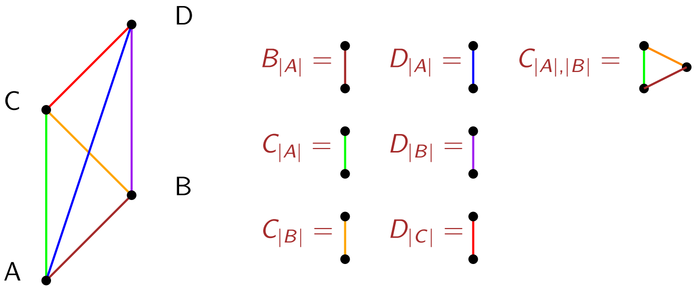
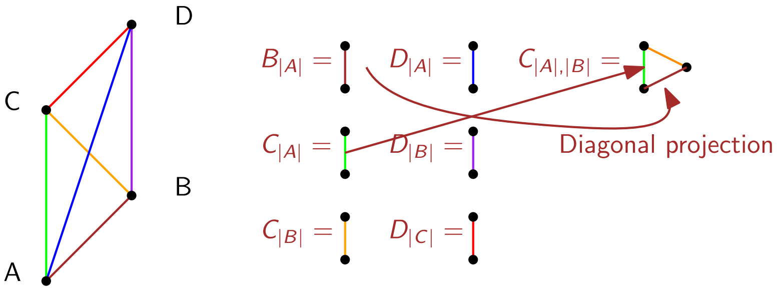


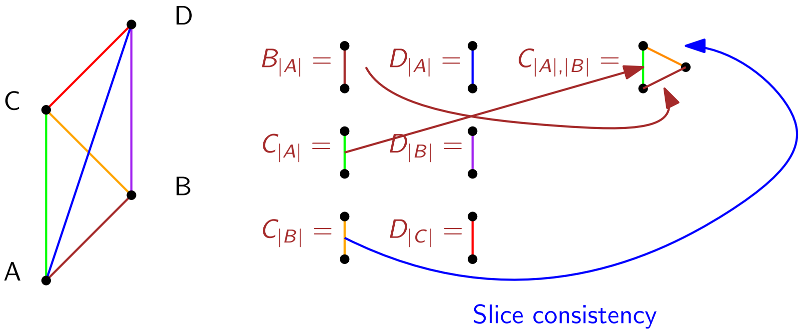
$\mathrm{Age}(\overline{\mathbf{G}}_{\mathbf{H}}^{c}) = \mathrm{Age}(\mathbf{H})$



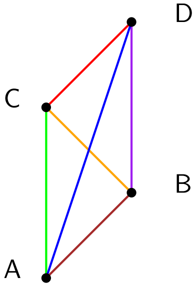
$\mathrm{Age}(\overline{\mathbf{G}}^c_{\mathbf{H}}) = \mathrm{Age}(\mathbf{H})$

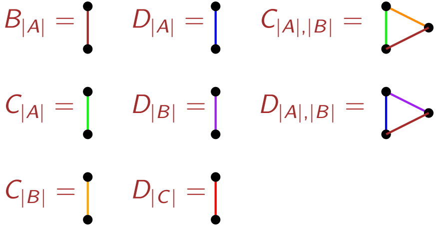


$\text{Age}(\overline{\mathbf{G}}^{c}_{\mathbf{H}}) = \text{Age}(\mathbf{H})$



Age($\overline{\mathbf{G}_\mathbf{H}}^c$) = Age($\mathbf{H}$)





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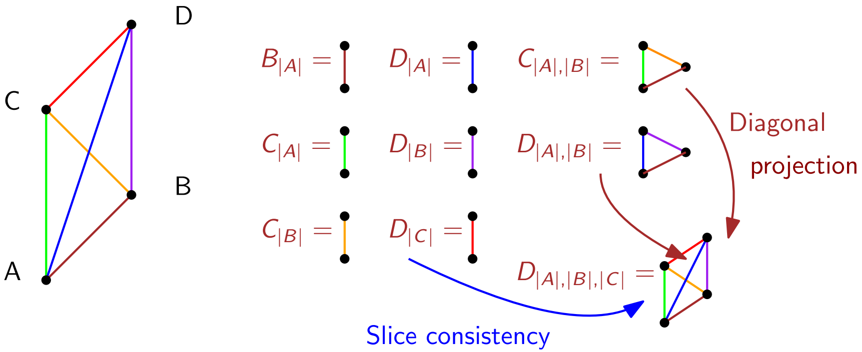

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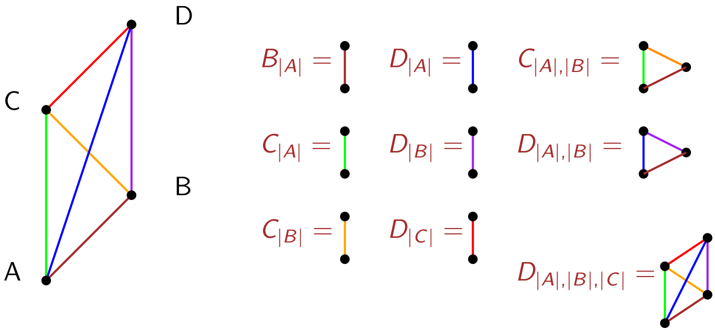

Recall that Σ be the class of all enumerated structures in $\mathcal{K}$ with at most c vertices.

Thank you for the attention

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- M. Balko, D. Chodounský, J.H., M. Konečný, L. Vena: Big Ramsey degrees of 3-uniform hypergraphs are finite, arXiv:2008.00268, 2020.
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