Unifying Themes in Ramsey Theory

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Ramsey classes and EPPA

Definition (Ramsey class, Reminder)

A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Definition (Extension property for partial automorphisms)

A class $\mathcal{C}$ of finite L-structures has extension property for partial automorphisms (EPPA or Hrushovski property) iff for every $\mathbf{A} \in \mathcal{C}$ there exists $\mathbf{B} \in \mathcal{C}$ containing $\mathbf{A}$ such that every partial automorphism of $\mathbf{A}$ extends to automorphism of $\mathbf{B}$

Partial automorphism is any isomorphism between two (induced) substructures.

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Example (Classes with EPPA)

1. Graphs (Hrushovski 1992)
2. Relational structures (Herwig 1998)
3. Classes described by finite forbidden homomorphisms (Herwig-lascar 2000)
4. Free amalgamation classes (Hodkinson and Otto 2003)
5. Metric spaces (Solecki 2005, Vershik 2008)
6. Generalizations and specializations of metric spaces (Conant 2015)

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Example (Classes with EPPA)

1. Graphs (Hrushovski 1992)
   Ramsey with free linear order (Nešetřil-Rödl 1977, Abramson-Harrington 1978)
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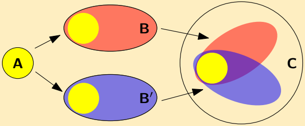
Strenghtenings of amalgamations

Definition (Amalgamation property of class $\mathcal{K}$)

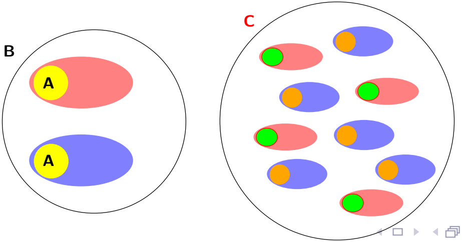


Observation (Nešetřil, 1980's)

Joint embedding + Ramsey $\implies$ amalgamation property

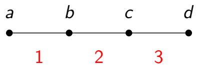
Observation

Joint embedding + EPPA $\implies$ amalgamation property



Common proof technique: EPPA for graphs (Herwig-Lascar)

The construction of EPPA-witness **A** (3-path):



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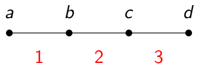

$$\psi(a) = \{1\} \qquad \psi(b) = \{1,2\}$$
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- Represent vertices by sets of edges they belong to

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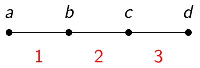

$$\psi(a) = \{1, 4\} \qquad \psi(b) = \{1, 2\}$$
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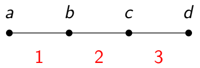

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1. Vertices: all 2-element subsets of $\{1, 2, 3, 4, 5\}$.
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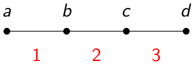

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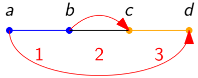

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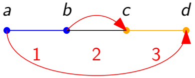

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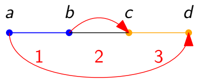

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- Partial automorphism of **A** "projects" to partial permutation of $\{1, 2, 3, 4, 5\}$.

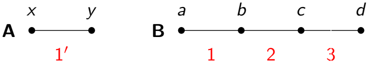
$$1 \mapsto 3, 4 \mapsto 5, 2 \mapsto 2$$

- Partial permutation extends to full permutation and induces automorphism of **B**

$$1 \mapsto 3, 4 \mapsto 5, 2 \mapsto 2, 3 \mapsto 1, 5 \mapsto 4$$

Common proof technique: Ramsey for graphs

The construction of **C** which is edge-Ramsey for an ordered 3-path:



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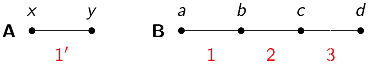

$$\begin{array}{l|ll}
\psi'(x) = \{\ 1'\} & \psi(a) = \{\ 1\} & \psi(b) = \{\ 1,2\} \\
\psi'(y) = \{\ 1'\} & \psi(c) = \{\ 2,3\} & \psi(d) = \{\ 3\}
\end{array}$$

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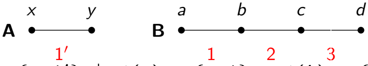

$$\psi'(x) = \{x, 1'\} \quad \psi(a) = \{a, 1\} \quad \psi(b) = \{b, 1, 2\}$$
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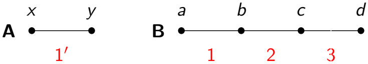

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- Represent vertices by sets of edges they belong to
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- Order vertices and order edges lexicographically: $\{x, y, 1'\}$, $\{a, b, c, d, 1, 2, 3\}$.

Construction of Ramsey graph $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$

By Graham-Rothschild thm. for $\Sigma = \{\circ\}$ get N s. t. for every coloring of 3-parametric subspaces of N-parametric space has monochromatic 7-parametric subspace.

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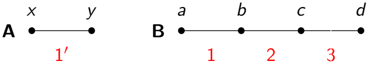

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Coloring of edges induces coloring of 3-parametric words and leads to monochromatic copy of 7-parametric word giving monochromatic copy of **B**.

		a	b	c	d	1	2	3
$\Psi(1)$	$=$	Λ_1	Λ_2	$\circ$	$\circ$	Λ_3	Λ_2	$\circ$
$\Psi(2)$	$=$	$\circ$	Λ_1	Λ_2	$\circ$	Λ_1	Λ_3	Λ_2
$\Psi(3)$	$=$	$\circ$	$\circ$	Λ_1	Λ_2	$\circ$	Λ_1	Λ_3

From graphs to other classes of structures

- Both constructions extends to relational structures in general but needs more care.
- Neither construction can handle classes with forbidden substructure and needs to be replaced by more involved tools:
  - the (Nešetřil-Rödl's) partite construction for Ramsey objects
  - Herwig-Lascar theorem for EPPA in full generality. Hodkinson-Otto and earlier Herwig's construction for irreducible substrucutres.
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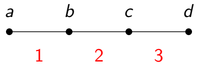
We show systematic proof of a strenghtening of Herwig-Lascar theorem.

New construction of EPPA for graphs



- Represent vertex $x \in A$ by function $\psi_x \colon A \setminus \{x\} \to \{0, 1\}$ that corresponds to given row of the asymmetric incidence matrix of **A**.

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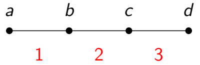

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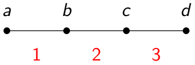

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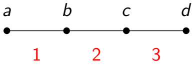

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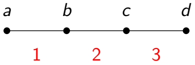

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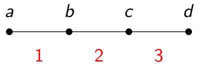

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- **B** is an EPPA witness of **A**.

New construction of EPPA for graphs

Construction of EPPA-witness B of A

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Embeding $\psi : \mathbf{A} \to \mathbf{B}$ is $\psi(x) = \psi_x$ s.t. $\psi_x(y) = I_{x,y}$ (row of assym. incidence matrix).

Theorem (H., Konečný, Nešetřil, 2018)

Every partial automorphism φ of $\psi(\mathbf{A})$ extends to automorphism $\mathbf{B}$.

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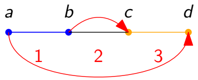
Proof.

- φ induces partial permutation of A which extends to full permutation $\hat{\varphi}(A)$.

□



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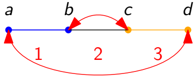

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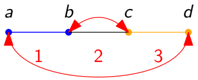

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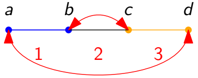
For all $y \in A \setminus \hat{\varphi}(x)$. □



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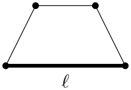
- Construction has Ramsey counterpart: Sauer's proof of finite big Ramsey degree for Rado graph:
  Rows of assymetric incidence matrix truncated at the diagonals $\iff$ vertices of the Milliken's tree with passing number representation of edges.

New construction of EPPA for metric spaces

We consider metric spaces to be complete edge-labeled graphs (or relational structures).

Lemma

Let **A** *be a metric space seen (complete) edge-labeled graph.*
If there is EPPA-witness **B** *(possibly with no distance defined for some pairs of vertices) which contains no induced non-metric cycles, then* **B** *can be completed to a metric space* **C** *which is EPPA-witeness of* **A**.



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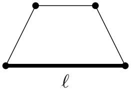

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C has same vertex set as **B** and distance of x, y in **C** is the length of shortest path connecting x and y in **B**. □

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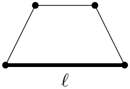
Main idea: Repeat the valuation trick to unwind cycles to “Möbius strips”



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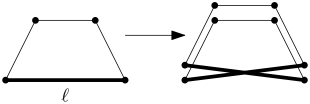

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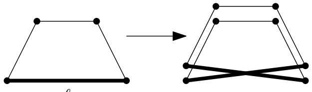

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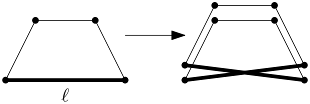

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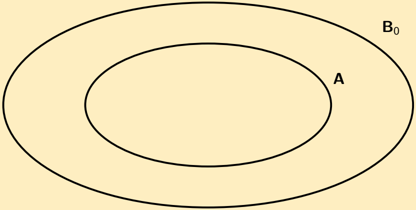

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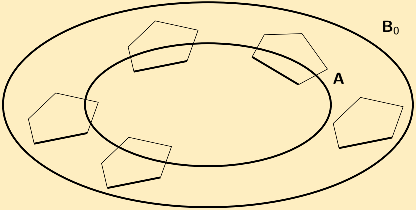

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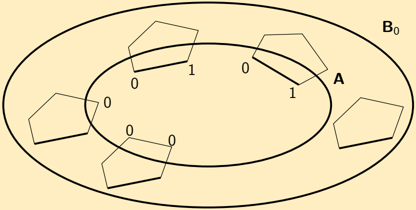

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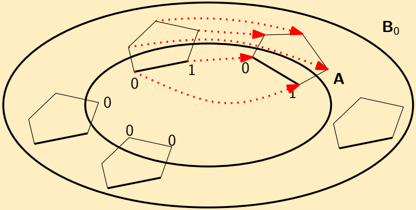

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Proof.

- Given metric space $\mathbf{A}$ construct EPPA-witness $\mathbf{B}_0$ (using construction for relational structures)
- Repeat Lemma N times to obtain $\mathbf{B}'$.

$$N = \left\lceil \frac{\max\{d_{\mathbf{A}}(x,y); x \neq y \in A\}}{\min\{d_{\mathbf{A}}(x,y); x \neq y \in A\}} \right\rceil$$

- Complete $\mathbf{B}'$ to $\mathbf{B}$ by the shortest path completion.

□

Irreducible structure faithful EPPA

Structure **C** is irreducible iff it is not free amalgamation of its two proper substructures.

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EPPA-witness **B** of **A** is irreducible-structure faithfull if for every irreducible substructure $\mathbf{C} \subseteq \mathbf{B}$ there exists automorphism $\varphi : \mathbf{B} \to \mathbf{B}$ such that $\varphi(\mathbf{C}) \subseteq \mathbf{A}$.

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- Substructure $\mathbf{C} \subseteq \mathbf{B}_0$ is bad if it is irreducible and there is no automorphism $\varphi : \mathbf{B}_0 \to \mathbf{B}_0$ such that $\varphi(\mathbf{C}_0) \subseteq \mathbf{A}_0$
- Given $x \in B$, $U(x)$ is the set of all bad substructures containing x.
- Given $x \in B$, x-valuation is $\chi_x : U(x) \to \mathbb{N}$ such that $\forall_{\mathbf{C} \in U(x)} : \chi_x(\mathbf{C}) \leq |C| - 1$.

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Herwig-Lascar theorem

- Given L-structures **A** and **B**, function $f : A \to B$ is homomorphism if for every $R \in L$, $(x_1, \dots x_a) \in R_{\mathbf{A}}$ it holds that $(f(x_1), \dots f(x_a)) \in R_{\mathbf{B}}$
- f is homomorhism-embedding if the restriction $f|_{\mathbf{C}}$ is an embedding whenever **C** is an irreducible substructure of **A**
- $\mathrm{Forb_{he}}(\mathcal{F})$ is the set of all structures **A** such that there is no $\mathbf{F} \in \mathcal{F}$ with homomorphism-embedding to **A**.

Theorem (Herwig-Lascar, 2000)

Let L be relational language, $\mathcal{F}$ be a finite family of finite L-structures. If there is possibly infinite "EPPA-witness" of $\mathbf{M} \in \mathrm{Forb_{he}}(\mathcal{F})$ *of* **A** *then there is finite EPPA-witness* $\mathbf{B} \in \mathrm{Forb_{he}}(\mathcal{F})$ *of* **A**.

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- $\mathrm{Forb_{he}}(\mathcal{F})$ is the set of all structures **A** such that there is no $\mathbf{F} \in \mathcal{F}$ with homomorphism-embedding to **A**.

Theorem (Herwig-Lascar, 2000)

Let L be relational language, $\mathcal{F}$ be a finite family of finite L-structures. If there is possibly infinite "EPPA-witness" of $\mathbf{M} \in \mathrm{Forb_{he}}(\mathcal{F})$ *of* **A** *then there is finite EPPA-witness* $\mathbf{B} \in \mathrm{Forb_{he}}(\mathcal{F})$ *of* **A**.

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If $\mathcal{F}$ is as in Herwig-Lascar theorem and **C** *constructed by series of free amalgamation of copies of* **A** *over their substructures, then* $\mathbf{C} \in \mathrm{Forb_{he}}(\mathcal{F})$.

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Construction of EPPA-witness B of A

- Fix **A**. WLOG assume that there is $E \in L$ and **A** has the property that for every $x \neq y \in A$ it holds that $(x, y) \in R_{\mathbf{E}}^{A}$.
- Construct EPPA-witness $\mathbf{B}_0$.

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Construction of EPPA-witness $\mathbf{B}$ of $\mathbf{A}$

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- Construct irreducible-structure faithful EPPA-witness $\mathbf{B}_1$ with homomorphism embedding to $\mathbf{B}_0$.
- Put $n = \max\{|F|; \mathbf{F} \in \mathcal{F}\}$, $N = \binom{n}{2}$.
- Consider every induced graph cycle in relation E bad, repeat N times the construction unwinding bad cycles to get $\mathbf{B}_1 \ldots \mathbf{B}_N$ such that $\mathbf{B}_i$ has homomorphism embedding to every $\mathbf{B}_j$, $j \leq i$. Put $\mathbf{B} = \mathbf{B}_N$.

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Proof.

- We constructed sequence $\mathbf{B}_1, \ldots \mathbf{B}_N$. Each $\mathbf{B}_i$ is an irreducible-structure faithful EPPA-witness of $\mathbf{A}$. For every $1 \leq i \leq N$ there is homomorphism-embedding $f_i : \mathbf{B}_i \to \mathbf{B}_{i-1}$ with property that every $f_i(\mathbf{C})$ is not isomorphic to $\mathbf{C}$ for every induced graph cycle in relation $E^{\mathbf{B}_i}$.

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- Consider $\mathbf{D} \subseteq \mathbf{B}_N$ with at most n vertices. One of the following holds
  1. $\mathbf{D}$ contains no induced cycles in relation E
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- By Lemma we know that $\mathbf{B}_N \in \mathrm{Forb_{he}}(\mathcal{F})$.

□

Symetric version of model-theoretic structures

- Let L be a language with relation symbols and function symbols each with arity denoted by $a(R)$ and $a(F)$.
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We consider Γ_L-structures which are essentially L-structures with the following definition of homomorphism:

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(a) $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \implies (f_A(x_1), f_A(x_2), \ldots, f_A(x_{a(R)})) \in f_L(R)_{\mathbf{B}}$, and,

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Notion of embedding, homomorphism-embedding, substructure, EPPA-witness and irreducible-structure faithfulness generalize naturally to this category.

A strenghtening of Herwig-Lascar theorem

Theorem (H., Konečný, Nešetřil, 2018)

Let L be a finite language with relations and unary functions equipped with a permutation group Γ_L. Let $\mathcal{F}$ be a finite family of finite Γ_L-structures.

If there exists a possibly infinite "EPPA-witness" $\mathbf{M} \in \mathrm{Forb_{he}}(\mathcal{F})$ *of* $\mathbf{A}$*, then there exists a finite structure* $\mathbf{B} \in \mathrm{Forb_{he}}(\mathcal{F})$ *which is an irreducible structure faithful coherent EPPA-witness of* $\mathbf{A}$*.*

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Proof follows same three steps using valuation constructions:

1. Construction of an EPPA-witness $\mathbf{B}_0$.
2. Construction of an irreducible-structure faithful EPPA-witness $\mathbf{B}_1$.
3. Construction of EPPA-witnesses $\mathbf{B}_2, \ldots \mathbf{B}_N$ unwinding cycles in copies.

Unary functions are added by simple "local covering" technique used earlier by Evans, H., Nevšetřil, 2017.

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Unary functions and language permutations are useful tools for structures with definable equivalences.

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- Free amalgamation classes of Γ_L structures are Ramsey
    - 2-orietnations of the Ω-categorical Hrushovski constructions have EPPA (this answers question of Evans, H., Nešetřil)
    - Two-graphs has EPPPA (this answers question of MacPherson and Siniora; coherency remain open)
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Open problems:

- Class of all structures with one binary function
- Class of all partial steiner systems
- Class with two equivalence classes on 2-tuples
- Class of all tournaments

Relationship to the structural Ramsey theory

- Our proof follows exactly the same structure as proof of the main result of:

  H., Nešetřil: All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)
- Resulting structural conditions for class being Ramsey or have EPPA are almost the same except for:
  - Ramsey classes are always ordered; EPPA classes never define order.
  - To obtain an EPPA one needs completion that is automorphism preserving (EPPA for tournaments remain open).
- Constructions are very similar to the construction used to show finite big Ramsey degree.

Thank you for the attention

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